Applications of Formal Verification

Formal Software Design: Modelling in Event-B

Bernhard Beckert · Mattias Ulbrich | SS 2019
Jean-Raymond Abrial: 
*Modelling in Event-B: System and Software Engineering*
Cambridge University Press, 2010

Jean-Raymond Abrial: 
*The B-Book: Assigning programs to meanings*
Cambridge University Press, 1996
Abstraction and Refinement –
Introduction
Late fault recovery is expensive

Late fault recovery is expensive

Goal: Detect faults here!

Reasons for system faults

- Systems are inherently complex
- Unconsidered situations, corner cases
- Ambiguous natural language requirements
- Component interplay
- …
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Abstraction

The only tool to master complexity is abstraction.

Cliff Jones
Abstraction and Refinement
Abstraction and Refinement

Abstraction

Concrete

Refinement

Abstraction
Abstraction and Refinement

Abstraction

Concrete

Refinement

Abstraction
Abstraction and Refinement

Abstraction

Concrete

Refinement
Abstraction and Refinement

Abstract

Refinement

Concrete

Abstraction
Abstraction

- reduce system complexity
- without removing important properties
- make the model susceptible to formal analysis

and the inverse

Refinement

- enrich abstract model with details
- introduce a new particular aspect
- iterative process: add complexity in a stepwise fashion
Abstraction in Engineering

Abstraction is an important tool in engineering

Established means of abstraction

- Mechanical engineering: BLUEPRINTS
- Electrical engineering: DATASHEETS
- CIRCUIT DIAGRAMS
- Architecture: FLOOR PLANS
- ...

Abstract descriptions remove unnecessary details, concentrate on one aspect
Datasheet – Abstraction

Extracts from datasheet for an IC with four NAND gates
Datasheet – Abstraction

Extracts from datasheet for an IC with four NAND gates

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Aspect **Behaviour**

refined to

Aspect **Geometry**

Beckert, Ulbrich – Applications of Formal Verification
Datasheet – Abstraction
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Aspect Behaviour
refined to

Aspect Geometry
refined to
Schematic Diagram vs. PCB Layout

Arduino™ UNO Rev3
Schematic Diagram vs. PCB Layout

Aspect
“Behaviour” preserved
Beck diagrams (1931)
Beck diagrams (1931)

Aspect “Route planning” is preserved
Property preservation

Abstraction with focus on particular aspect
System properties w.r.t. that aspect must also hold in the abstraction.

Refinement with focus on particular aspect
Properties of abstract model w.r.t. that aspect must be inherited by the refined model.

Examples:

- Abstraction: “The shortest tube travel from Liverpool St. to Westminster has 8 stops and 2 changes.”
- Refinement: Abstract: Input “$a = 1$” gives output “$b = 1$”
  Concrete: High voltage on pin A gives high voltage on pin B
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That’s what we will formally prove in the next sections.

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“Conceptual” vs “Technical” Abstraction

Two areas of abstraction and refinement in formal methods:

Conceptual abstraction

Abstraction as a technique

That's what we will look into in the next sections.
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Event-B –
Introduction
Event-B

- EventB is a formalism for modelling and reasoning about discrete systems.
  - for their structure (how can their state be described) and
  - for their behaviour (how can the evolution of their state be described)

- Models are formulated using set theory

- Event-based evolution of the original B Method

- Tool-support:
  - RODIN – deductive verification, theorem prover: proofs
  - Pro-B – model checking, animator: counterexamples
Central Concepts

- **Variables and Events**
  - *Variables* model the current state within the state space.
  - *Events* describe operations to model the system behaviour.

- **Invariants**
  - Properties to be maintained by the system.
  - Formal proof obligations to show that the invariants are maintained.

- **Refinement**
  - Behaviour of refining model is compatible with abstract model.
  - Formal proof obligation to show that.
  - Hence, invariants of abstract model are inherited by concrete model.
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- Hence, invariants of abstract model are inherited by concrete model
Event-B models consist of contexts and machines:

**Contexts**
Static, rigid, constant parts that do not change over time.

**Machines**
Dynamic, volatile, evolving parts that do change over time.
# Contexts and Machines

Event-B models consist of **contexts and machines**:

## Contexts
- *Carrier sets* (ground types, universes, “urelements”)
- *Constants* (state-independent symbols, rigid symbols)
- *Axioms* (formulas valid by stipulation)
- *Theorems* (formulas proved valid)

## Machines
- *Context references* (which symbols are available)
- *Variables* (state-dependent symbols, non-rigid symbols, program variables)
- *Invariants* (formulas true in every reachable system state)
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Students and Exams – Requirements

R1 Every **student** must take a final exam in a **subject** of their choice.

R2 They can have **attempts** without yet failing or passing.

R3 Eventually they can **pass** or **fail**, but **never both**.

→ Identify the **context**, the **state** and the **events** according to the requirements **R1–R3**.
Introduction by Example

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CONTEXT ExamCtxt

...
CONTEXT ExamCtxt

SETS

STUDENT // see requirement R1
SUBJECT
Exam Context

CONTEXT ExamCtxt

SETS
STUDENT // see requirement R1
SUBJECT

CONSTANTS
maths physics siblings
Exam Context

CONTEXT ExamCtxt

SETS

STUDENT // see requirement R1
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CONSTANTS

maths physics siblings

AXIOMS

maths ∈ SUBJECT // type of variables
physics ∈ SUBJECT
CONTEXT ExamCtxt

SETS
STUDENT // see requirement R1
SUBJECT

CONSTANTS
maths physics siblings

AXIOMS
maths ∈ SUBJECT // type of variables
physics ∈ SUBJECT
maths ≠ physics // constants could have same value
Exam Context

CONTEXT ExamCtxt

SETS
STUDENT // see requirement R1
SUBJECT

CONSTANTS
maths  physics  siblings

AXIOMS
maths ∈ SUBJECT // type of variables
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siblings ⊆ STUDENT × STUDENT // function type
CONTEXT ExamCtxt

SETS
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maths ∈ SUBJECT // type of variables
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maths ≠ physics // constants could have same value
siblings ⊆ STUDENT × STUDENT // function type
∀ s · s ∈ STUDENT ⇒ (s → s) ∉ siblings // irreflexive
// ...
MACHINE ExamAbstract
MACHINE ExamAbstract
SEES ExamCtxt
MACHINE ExamAbstract
SEES ExamCtxt

VARIABLES
  passed  failed
MACHINE ExamAbstract
SEES ExamCtxt

VARIABLES
  passed    failed

INVARIANTS
  passed ⊆ STUDENT    failed ⊆ STUDENT
Exam Machine

MACHINE ExamAbstract
SEES ExamCntxt

VARIABLES
\[ \text{passed} \quad \text{failed} \]

INVARIANTS
\[ \text{passed} \subseteq \text{STUDENT} \quad \text{failed} \subseteq \text{STUDENT} \]
\[ \text{passed} \cap \text{failed} = \emptyset \quad // \text{R3} \]
MACHINE ExamAbstract
SEES ExamCtxt

VARIABLES
  passed    failed

INVARIANTS
  passed ⊆ STUDENT    failed ⊆ STUDENT
  passed ∩ failed = ∅  // R3

EVENTS
  INITIALISATION ≡ ...
  ATTEMPT_exam ≡ ...  // R2
  pass_exam ≡ ...      // R3
  fail_exam ≡ ...      // R3
Exam Machine (2)

MACHINE ExamAbstract
VARIABLES passed failed . . .

EVENTS
INITIALISATION ≜
   failed := ∅
   passed := ∅
MACHINE ExamAbstract
VARIABLES passed failed . . .

EVENTS
INITIALISATION ≜
  failed := \emptyset
  passed := \emptyset

PASSEXAM ≜
  ANY s grade
  WHERE s ∈ STUDENT ∧ grade ≤ 4
  THEN passed := passed ∪ \{ s \}
MACHINE ExamAbstract
VARIABLES passed failed . . .

EVENTS
INITIALISATION \equiv
\begin{align*}
\text{failed} & \equiv \emptyset \\
\text{passed} & \equiv \emptyset
\end{align*}

PASS\textsc{Exam} \equiv
\begin{align*}
\text{ANY } s \text{ grade} \\
\text{WHERE } s \in \text{STUDENT} \land \text{grade} \leq 4 \\
\text{THEN } \text{passed} & \equiv \text{passed} \cup \{s\}
\end{align*}

FAIL\textsc{Exam} \equiv
\begin{align*}
\text{ANY } s \text{ grade} \\
\text{WHERE } s \in \text{STUDENT} \land \text{grade} > 4 \\
\text{THEN } \text{failed} & \equiv \text{failed} \cup \{s\}
\end{align*}
MACHINE ExamAbstract
VARIABLES passed failed
INVARIANTS passed \cap failed = \emptyset \quad \ldots

EVENTS
PASS_EXAM \triangleq \\
\text{ANY } s \text{ grade} \\
\text{WHERE } s \in \text{STUDENT} \land \text{grade} \leq 4 \\
\text{THEN } passed := passed \cup \{s\}

FAIL_EXAM \triangleq \\
\text{ANY } s \text{ grade} \\
\text{WHERE } s \in \text{STUDENT} \land \text{grade} > 4 \\
\text{THEN } failed := failed \cup \{s\}
MACHINE ExamAbstract
VARIABLES passed failed
INVARIANTS passed \cap failed = \emptyset \ldots

EVENTS
PASS\textsc{Exam} \triangleq
\begin{align*}
\text{ANY } s \text{ grade} \\
\text{WHERE } s \in \text{STUDENT} \setminus (\text{failed} \cup \text{passed}) \land \text{grade} \leq 4 \\
\text{THEN } passed := passed \cup \{s\}
\end{align*}

FAIL\textsc{Exam} \triangleq
\begin{align*}
\text{ANY } s \text{ grade} \\
\text{WHERE } s \in \text{STUDENT} \setminus (\text{failed} \cup \text{passed}) \land \text{grade} > 4 \\
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\end{align*}
Underspecified model

**EVENTS**

**PASS**\textsc{Exam} \define \[
\text{ANY } s \text{ grade WHERE } \text{grade} \leq 4 \land s \in \ldots \text{ THEN } \text{passed} := \text{passed} \cup \{s\}
\]

**FAIL**\textsc{Exam} \define \[
\text{ANY } s \text{ grade WHERE } \text{grade} > 4 \land s \in \ldots \text{ THEN } \text{failed} := \text{failed} \cup \{s\}
\]

**ATTEMPT**\textsc{Exam} \define \[
\text{ANY } s \text{ grade WHERE } \text{grade} \in \mathbb{N} \land s \in \ldots \text{ THEN } \text{skip}
\]
Underspecified model

EVENTS
PASS\text{EXAM} \triangleq
\text{ANY } s \text{ grade WHERE } \text{grade} \leq 4 \land s \in \ldots
\text{THEN } \text{passed} \equiv \text{passed} \cup \{ s \}

FAIL\text{EXAM} \triangleq
\text{ANY } s \text{ grade WHERE } \text{grade} > 4 \land s \in \ldots
\text{THEN } \text{failed} \equiv \text{failed} \cup \{ s \}

ATTEMPT\text{EXAM} \triangleq
\text{ANY } s \text{ grade WHERE } \text{grade} \in \mathbb{N} \land s \in \ldots
\text{THEN } \text{skip}

Additional requirement

R4 Any student may attempt the exam three times and ultimately fails if the fourth attempt is unsuccessful.
MACHINE RefinedExams REFINES ExamsAbstract
MACHINE RefinedExams REFINES ExamsAbstract

VARIABLES passed attempts
Refinement Exams (1)

MACHINE RefinedExams REFINES ExamsAbstract
VARIABLES passed attempts
INVARIANTS

\[
\text{attempts} \in \text{STUDENT} \rightarrow \mathbb{N} \quad // \quad \text{typing for attempts}
\]

\[
\text{failed} = \{ s \cdot \text{attempts}(s) = 4 \} \quad // \quad \text{coupling invariant}
\]
MACHINE RefinedExams REFINES ExamsAbstract
VARIABLES passed attempts
INVARIANTS
\[
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\]
EVENTS
INITIALISATION \models \text{REFINES} \text{INITIALISATION}
\[
\text{passed} := \emptyset
\]
\[
\text{attempts} := \{ s \cdot s \in \text{STUDENT} \mid (s \mapsto 0) \}
\]
MACHINE RefinedExams REFINES ExamsAbstract
VARIABLES passed attempts

INVARIANTS
\[ \text{attempts} \in \text{STUDENT} \rightarrow \mathbb{N} \quad // \text{typing for attempts} \]
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EVENTS

INITIALISATION \[ \cong \text{REFINES} \text{ INITIALISATION} \]
\[ \text{passed} := \emptyset \]
\[ \text{attempts} := \{ s \cdot s \in \text{STUDENT} \mid (s \mapsto 0) \} \]

EXAM\text{ULTIMATEFAIL} \[ \cong \text{REFINES} \text{ EXAMFAIL} \ldots \]
EXAM\text{MISSED} \[ \cong \text{REFINES} \text{ EXAMATTEMPT} \ldots \]
EXAM\text{PASSED} \[ \cong \text{REFINES} \text{ EXAMPASSED} \ldots \]
... EVENTS

\textsc{examUltimateFail} \triangleright= \textsc{refines examFail}
\begin{align*}
\text{ANY } s & \text{ grade} \\
\text{WHERE } ... \land \text{ grade} > 4 \land \text{ attempts}(s) = 3 \\
\text{THEN} \\
\text{attempts}(s) & := \text{attempts}(s) + 1
\end{align*}

\textsc{examMissed} \triangleright= \textsc{refines examAttempt}
\begin{align*}
\text{ANY } s & \text{ grade} \\
\text{WHERE } ... \land \text{ grade} > 4 \land \text{ attempts}(s) < 3 \\
\text{THEN} \\
\text{attempts}(s) & := \text{attempts}(s) + 1
\end{align*}

...
This refinement takes now also R4 into account.

Refinement preserves invariants

\[\begin{align*}
\text{Every possible event of } & \textit{RefinedExams} \text{ is a possible event in } \textit{ExamsAbstract} \\
\Rightarrow & \text{Every invariant of } \textit{ExamsAbstract} \text{ is also an invariant of } \textit{RefinedExams}
\end{align*}\]

We will come back to this more formally ...
Set Theory – Equipment for formal modelling
Set theory – a universal modelling language

Not only used in Event-B.

<table>
<thead>
<tr>
<th>Set theory also used for modelling in ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>▪ Z</td>
</tr>
<tr>
<td>▪ Object-Z, Z++</td>
</tr>
<tr>
<td>▪ (classical) B</td>
</tr>
<tr>
<td>▪ Event-B</td>
</tr>
<tr>
<td>▪ Alloy</td>
</tr>
<tr>
<td>▪ ...</td>
</tr>
</tbody>
</table>
Every term in Event-B has a unique type.

Types are *part of the syntax* of Event-B and some expressions are syntactically forbidden:

\[ \text{maths} \in \text{failed} \] is syntactically invalid.

(remember: \( \text{math} \in \text{SUBJECT}, \text{failed} \subseteq \text{STUDENT} \))

“You can’t compare apples and oranges.”
Set Theory

Formal language in Event-B models

Typed **First Order Set Theory** with Additional Theories

- sets are objects in the logic
- first order axioms define the semantics of sets
- quantification over sets is allowed
- quantification over predicates, functions is not allowed
- (Foundation is a typed Zermelo-Fraenkel axiomatisation)
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- quantification over sets is allowed
- quantification over predicates, functions is not allowed
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There are additional theories with fixed semantics

- integers

- more theories (datatypes) can be added by user (an extension to the system)
Types

1. **BOOL** and \( \mathbb{Z} \) are types

2. Every carrier set declared in a **CONTEXT** is a type.

3. If \( T \) is a type then \( \mathcal{P}(T) \) is a type.
   
   **Semantics:** \( \mathcal{P}(T) \) is the set of all subsets of \( T \) (powerset).

4. If \( T_1, T_2 \) are types then \( T_1 \times T_2 \) is a type.
   
   **Semantics:** \( T_1 \times T_2 \) is the set of all ordered pairs \((a, b)\) with \( a \in T_1 \) and \( b \in T_2 \) (Cartesian product).

Every expression \( E \) has a unique type \( \tau(E) \).
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Set theory needs not be typed: Everything can be viewed as a set.

Reasons to introduce types:
- some specification errors may be detected as syntax errors (even before the verification has started)
- avoid Russell’s paradox
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Assume that the expression \( \{ s \mid \phi \} \) for any formula \( \phi \) denotes a set. Let \( R := \{ s \mid s \notin s \} \).
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One observes: \( R \in R \iff R \not\in R \) 

\[ \downarrow \]
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**Russell’s paradox**
Assume that the expression \( \{ s \mid \phi \} \) for any formula \( \phi \) denotes a set. Let \( R := \{ s \mid s \not\in s \} \). Not allowed with types.
One observes: \( R \in R \iff R \not\in R \)

(This crushed naive set theory in early 1900s.)
Sets

Constructors for sets:
- empty set $\emptyset : \mathcal{P}(S)$
Sets

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- powerset $\mathcal{P}(\cdot) : \mathcal{P}(S) \rightarrow \mathcal{P}(\mathcal{P}(S))$
  example: $\mathcal{P}({1, 2}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} : \mathcal{P}(\mathbb{Z})$
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  example: $\mathcal{P}\{\{1, 2\}\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} : \mathcal{P}(\mathbb{Z})$
- product $\times : \mathcal{P}(S) \times \mathcal{P}(T) \to \mathcal{P}(S \times T)$
  example: $BOOL \times \{1\} = \{\{true, 1\}, \{false, 1\}\} : \mathcal{P}(BOOL \times \mathbb{Z})$
Constructors for sets:

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- product $\cdot \times \cdot : \mathcal{P}(S) \times \mathcal{P}(T) \rightarrow \mathcal{P}(S \times T)$
  example: $\text{BOOL} \times \{1\} = \{\{\text{true}, 1\}, \{\text{false}, 1\}\} : \mathcal{P}(\text{BOOL} \times \mathbb{Z})$
- set comprehension $\{x \cdot \varphi \mid e\}$
  formula $\varphi$, term $e : T$, result of type $\mathcal{P}(T)$
  example: $\{x \cdot x \geq 2 \mid x \ast x\} = \{4, 9, 16, \ldots\}$
Relations

- Relations are sets of pairs (tuples).
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- Domain of a relation $\text{dom}(R)$
  \[ \text{dom}(R) = \{ x, y \cdot (x \leftrightarrow y) \in R \mid x \} \]
  example: $\text{dom}(E_1 \times E_2) = E_1$
Relations

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- Pairs \((E_1 \mapsto E_2) : \tau(E_1) \times \tau(E_2)\)

- Domain of a relation \( dom(R) \)
  \( dom(R) = \{x, y \cdot (x \mapsto y) \in R \mid x\} \)
  example: \( dom(E_1 \times E_2) = E_1 \)

- Range of a relation \( ran(R) \)
  \( ran(R) = \{x, y \cdot (x \mapsto y) \in R \mid y\} \)
  example: \( ran(E_1 \times E_2) = E_2 \)
Relations

- Relations are sets of pairs (tuples).
- All relations: $E_1 \leftrightarrow E_2 := P(E_1 \times E_2)$
- Pairs $(E_1 \leftrightarrow E_2) : \tau(E_1) \times \tau(E_2)$
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  \text{dom}(R) = \{ x, y \cdot (x \leftrightarrow y) \in R \mid x \}\]
  example: $\text{dom}(E_1 \times E_2) = E_1$
- Range of a relation $\text{ran}(R)$
  \[
  \text{ran}(R) = \{ x, y \cdot (x \leftrightarrow y) \in R \mid y \}\]
  example: $\text{ran}(E_1 \times E_2) = E_2$
- can be nested: $(E_1 \leftrightarrow E_2) \leftrightarrow E_3$ for a ternary relation etc.
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- **Domain** of a relation $dom(R)$
  
  $dom(R) = \{ x, y \cdot (x \leftrightarrow y) \in R \mid x \}$
  
  example: $dom(E_1 \times E_2) = E_1$ if $E_2 \neq \emptyset$

- **Range** of a relation $ran(R)$
  
  $ran(R) = \{ x, y \cdot (x \leftrightarrow y) \in R \mid y \}$
  
  example: $ran(E_1 \times E_2) = E_2$ if $E_1 \neq \emptyset$

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Kinds of relations

- All relations $E_1 \leftrightarrow E_2$

![Diagram showing relations with domain (dom) and range (ran)]
Kinds of relations

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- All surjections $E_1 \leftrightarrow E_2$ \hspace{1cm} (\text{ran}(R) = E_2)
Kinds of relations

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- All total relations $E_1 \leftrightarrow E_2 \quad (\text{dom}(R) = E_1)$
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- All total surjections $E_1 \leftrightarrow E_2$
Functional relations

Observation

Every function \( f \in A \rightarrow B \) can be understood as the relation
\[
\{ x \cdot x \in A \mid x \mapsto f(x) \} \in A \leftrightarrow B
\]

- Partial functions \( E_1 \mapsto E_2 \subseteq E_1 \leftrightarrow E_2 \)
  \((\forall x, y, z \cdot x \mapsto y \in R \land x \mapsto z \in R \Rightarrow y = z) \) (*)

\[ R \quad \text{dom} \quad \text{ran} \]
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- Total functions $E_1 \rightarrow E_2$
  \[
  E_1 \rightarrow E_2 = (E_1 \mapsto E_2) \cap (E_1 \leftrightarrow E_2)
  \]
  (both partial function and total relation)
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- Injections \( E_1 \leftrightarrow E_2 \)
  \[
  (*) \land (\forall x, y, z \cdot x \mapsto z \in R \land y \mapsto z \in R \Rightarrow x = y)
  \]
Intersection of relation classes give new classes:

- Total injections \( E_1 \leftrightarrow E_2 = (E_1 \rightarrow E_2) \cap (E_1 \leftrightarrow E_2) \)
- Partial surjections \( E_1 \leftrightarrow E_2 = (E_1 \rightarrow E_2) \cap (E_1 \leftrightarrow E_2) \)
- Total surjections \( E_1 \rightarrow E_2 = (E_1 \rightarrow E_2) \cap (E_1 \leftrightarrow E_2) \)
- Bijections \( E_1 \leftrightarrow E_2 = (E_1 \rightarrow E_2) \cap (E_1 \leftrightarrow E_2) \)
Example: File system

CONTEXT FileSystemCtx

\begin{align*}
\text{files} & \subseteq \text{OBJECT} \\
\text{dirs} & \subseteq \text{OBJECT} \\
\text{root} & \in \text{dirs} \\
\text{files} \cap \text{dirs} & = \emptyset
\end{align*}
Example: File system

CONTEXT FileSystemCtx
SETS OBJECT

CONSTANTS files, dirs, root

AXIOMS
files ⊆ OBJECT, dirs ⊆ OBJECT, root ∈ dirs, files ∩ dirs = ∅
Example: File system

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Example: File system

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      tree $\in$ dirs $\leftrightarrow$ (files $\cup$ dirs)
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    tree $\in$ dirs $\iff$ (files $\cup$ dirs)
    // most directories (but root) have a parent directory :
    parent $\in$ dirs $\mapsto$ dirs
Example: File system

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root ∈ dirs, files ∩ dirs = ∅

MACHINE FileSystem SEES FileSystemCtx
VARIABLES tree, parent
INVARIANTS
  tree ∈ dirs ↔ (files ∪ dirs)
  // most directories (but root) have a parent directory:
  parent ∈ dirs → dirs
  // more precise
  parent ∈ (dirs \ {root}) → dirs
Relational operations

- Relational application ·[·] : \( \mathcal{P}(S \times T) \times \mathcal{P}(S) \rightarrow \mathcal{P}(T) \)

\[
R[A] = \{ x, y \cdot x \mapsto y \in R \land x \in A \mid y \}\]

Beckert, Ulbrich – Applications of Formal Verification
SS 2019 42/96
**Relational operations**

- **Relational application** \( \cdot \left[ \cdot \right] : \mathcal{P}(S \times T) \times \mathcal{P}(S) \to \mathcal{P}(T) \)
  
  \[ R[A] = \{ x, y \cdot x \mapsto y \in R \land x \in A \mid y \} \]

- **Problem:** What if \( R[A] \) is not a one-element set?
  
  **Solution:** Well-definedness needs to be proved.

1. \( f \in S \mapsto \to T \) (not an arbitrary relation in \( S \leftrightarrow T \))
2. \( e \in \text{dom}(f) \) every time a functional application is used.

Beckert, Ulbrich – Applications of Formal Verification

SS 2019
Relational operations

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  \]

- **Functional application** \( \cdot (\cdot) : \mathcal{P}(S \times T) \times S \to T \)
  \[
  x = f(e) \iff e \mapsto x \in f \quad \{ f(e) \} = f[\{e\}] 
  \]
Relational operations

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  \[ x = f(e) \iff e \mapsto x \in f \quad \{ f(e) \} = f[e] \]

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everytime a functional application is used.
Restrictions

Concept

Limit the domain or range of a relation to a subset.
Restrictions

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Limit the domain or range of a relation to a subset.

\[
\begin{align*}
A \triangleleft - R &= \{ x, y \cdot x \mapsto y \in R \land x \in A \} \subseteq R \\
R \setminus - A &= \{ x, y \cdot x \mapsto y \in R \land y \notin A \} \subseteq R
\end{align*}
\]
Restrictions

Concept

Limit the domain or range of a relation to a subset.
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Concept

Limit the domain or range of a relation to a subset.

\[ A \mathrel{\triangleleft} R := \{ x, y : x \mapsto y \in R \land x \in A \mid x \mapsto y \} \subseteq R \]

\[ A \mathrel{\triangleright} R := \{ x, y : x \mapsto y \in R \land x \notin A \mid x \mapsto y \} \subseteq R \]

\[ R \mathrel{\triangleleft} B := \{ x, y : x \mapsto y \in R \land y \in B \mid x \mapsto y \} \subseteq R \]

\[ R \mathrel{\triangleright} B := \{ x, y : x \mapsto y \in R \land y \notin B \mid x \mapsto y \} \subseteq R \]
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\[ R \triangleright B \] := \{ x, y : x \mapsto y \in R \land y \in B \land x \mapsto y \} \subseteq R

Relational application: \( R[A] = \text{ran}(A \triangleleft R) \)
Override

\[ R \triangleleft S := ((\text{dom } S) \triangleleft R) \cup S \]

\[ x \mapsto y \in R \triangleleft S \iff \begin{cases} x \mapsto y \in S & \text{if } x \in \text{dom}(S) \\ x \mapsto y \in R & \text{if } x \not\in \text{dom}(S) \end{cases} \]

- “Clear” \( \text{dom}(S) \) in \( R \) and “replace” by \( S \).
Override

\[ R \leftarrow S := ((\text{dom } S) \leftarrow R) \cup S \]

\[ x \mapsto y \in R \leftarrow S \iff \begin{cases} x \mapsto y \in S & \text{if } x \in \text{dom}(S) \\ x \mapsto y \in R & \text{if } x \notin \text{dom}(S) \end{cases} \]

- “Clear” \text{dom}(S) in \( R \) and “replace” by \( S \).
- Special case: \( f \in A \rightarrow B, g \in A \rightarrow B \) implies \( f \leftarrow g \in A \rightarrow B \)
Override

\[ R \leftarrow S := ((\text{dom } S) \leftarrow R) \cup S \]

\[ x \mapsto y \in R \leftarrow S \iff \begin{cases} x \mapsto y \in S & \text{if } x \in \text{dom}(S) \\ x \mapsto y \in R & \text{if } x \notin \text{dom}(S) \end{cases} \]

- “Clear” \( \text{dom}(S) \) in \( R \) and “replace” by \( S \).
- Special case: \( f \in A \rightarrow B, g \in A \rightarrow B \) implies \( f \leftarrow g \in A \rightarrow B \)
- \( f \leftarrow \{x \mapsto y\} \) updates function \( f \) in one place \( x \)
Override

\[ R \triangleleft S := ((\text{dom } S) \triangleleft R) \cup S \]

\[ x \mapsto y \in R \triangleleft S \iff \begin{cases} x \mapsto y \in S & \text{if } x \in \text{dom}(S) \\ x \mapsto y \in R & \text{if } x \not\in \text{dom}(S) \end{cases} \]

- “Clear” \( \text{dom}(S) \) in \( R \) and “replace” by \( S \).
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- \( f \triangleleft \{x \mapsto y\} \) updates function \( f \) in one place \( x \)
- Caution: \( \triangleleft \) and \( \triangleleft \) are different symbols
Override

\[ R \leftarrow S := ((\text{dom } S) \leftarrow R) \cup S \]

\[ x \mapsto y \in R \leftarrow S \iff \begin{cases} 
  x \mapsto y \in S & \text{if } x \in \text{dom}(S) \\
  x \mapsto y \in R & \text{if } x \notin \text{dom}(S) 
\end{cases} \]

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- Syntax sometimes \( \oplus \) instead of \( \leftarrow \)
Override

\[ R \leftarrow S := ((\text{dom } S) \leftarrow R) \cup S \]

\[ x \mapsto y \in R \leftarrow S \iff \begin{cases} 
  x \mapsto y \in S & \text{if } x \in \text{dom}(S) \\
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\end{cases} \]

- “Clear” \( \text{dom}(S) \) in \( R \) and “replace” by \( S \).
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- \( f \leftarrow \{x \mapsto y\} \) updates function \( f \) in one place \( x \)

- Caution: \( \leftarrow \) and \( \leftarrow \) are different symbols
- Syntax sometimes \( \oplus \) instead of \( \leftarrow \)
- Compare *Updates* in Dynamic Logic for KeY.
Forward composition

\[ x \mapsto y \in R ; S \iff \exists z \cdot x \mapsto z \in R \land z \mapsto y \in S \]

\( x \mapsto y \) is in the composition \( R ; S \) if there is a transmitting element \( z \) with both \( x \mapsto z \in R \) and \( z \mapsto y \in S \).
\[ x \mapsto y \in R ; S \iff \exists z \cdot x \mapsto z \in R \land z \mapsto y \in S \]

\( x \mapsto y \) is in the composition \( R ; S \) if there is a transmitting element \( z \) with both \( x \mapsto z \in R \) and \( z \mapsto y \in S \).
Forward composition

\[ x \mapsto y \in R ; S \iff \exists z \cdot x \mapsto z \in R \land z \mapsto y \in S \]

\(x \mapsto y\) is in the composition \(R ; S\) if there is a transmitting element \(z\) with both \(x \mapsto z \in R\) and \(z \mapsto y \in S\).

(There is also backward composition \(R \circ S = S ; R\))
Example: File system

CONTEXT FileSystemCtx
SETS OBJECT
CONSTANTS files, dirs, root
AXIOMS files ⊆ OBJECT, dirs ⊆ OBJECT,
root ∈ dirs, files ∩ dirs = ∅

MACHINE FileSystem SEES FileSystemCtx
VARIABLES tree, depth
INVARIANTS
  tree ∈ dirs ↔ (files ∪ dirs) ∧ depth ∈ dirs → ℕ ∧
Example: File system

CONTEXT FileSystemCtx
SETS OBJECT
CONSTANTS files, dirs, root
AXIOMS files ⊆ OBJECT, dirs ⊆ OBJECT, 
    root ∈ dirs, files ∩ dirs = ∅

MACHINE FileSystem SEES FileSystemCtx
VARIABLES tree, depth
INVARIANTS
    tree ∈ dirs ↔ (files ∪ dirs) ∧ depth ∈ dirs → N ∧ 
    ∀d · ((depth(d) > 0 ⇒ depth[tree[{d}]] = {depth(d) − 1}) 
    ∧ (depth(d) = 0 ⇒ {d} ⊲ tree ⊳ files = ∅))
Event-B –
Events
The symbols in context can be used in \textit{inv} even if not mentioned explicitly.
EVENT $M$

// the following are the parameters,
// the input signals, nondeterministic choices
ANY $prms$

// the preconditions, conditions on the input values
WHERE $guard(vars, prms)$

// evolution of the program variables when the event “fires”
THEN
   actions
END

There is one more construct (WITH) that we omit here.
Deterministic actions

- "Assignment" $x := t$
- Variable $x$ and term $t$ must have same type ($\tau(t) = \tau(x)$)
- After event, $x$ has value of expression $t$
**Actions** (Generalised Substitutions)

---

**Deterministic actions**

- “Assignment” $x := t$
- Variable $x$ and term $t$ must have same type ($\tau(t) = \tau(x)$)
- After event, $x$ has value of expression $t$
Actions (Generalised Substitutions)

Deterministic actions

- “Assignment” \( x := t \)
- Variable \( x \) and term \( t \) must have same type \( (\tau(t) = \tau(x)) \)
- After event, \( x \) has value of expression \( t \)
**Deterministic actions**

- "Assignment" $x := t$
- Variable $x$ and term $t$ must have same type ($\tau(t) = \tau(x)$)
- After event, $x$ has value of expression $t$
Deterministic actions

- “Assignment” \( x := t \)
- Variable \( x \) and term \( t \) must have same type (\( \tau(t) = \tau(x) \))
- After event, \( x \) has value of expression \( t \)

Example:

```plaintext
THEN
  x := y
  y := x
END  // swaps values of variables \( x, y \).
```

Unmentioned variable \( z \) does not change.

Remember the updates in KeY: \( \{ x := y || y := x \} \) has same effects.
**Actions** (Generalised Substitutions)

### Nondeterministic actions

$x : | \varphi$ means “choose $x$ such that $\varphi$”

- Actions can have more than one resolution
- $\varphi$ is called the before-after-predicate (BAP)
- Variables without tick: before-state
- Variables with tick: after-state.

Example:

$x, y : | x' = y' \land y' > y$  
After the action $x$ and $y$ are equal and $y$ is strictly greater than before the action.
Nondeterministic actions

\[ x : \varphi \quad \text{means} \quad \text{“choose } x \text{ such that } \varphi \text{”} \]

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Actions (Generalised Substitutions)

Nondeterministic actions

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Example:

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After the action \( x \) and \( y \) are equal and \( y \) is strictly greater than before the action.
**Actions** (Generalised Substitutions)

Nondeterministic actions

\[ x : \varphi \] means “choose \( x \) such that \( \varphi \)”

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Example: \( x, y : x' = y' \land y' > y \) After the action \( x \) and \( y \) are equal and \( y \) is strictly greater than before the action.
**Actions (Generalised Substitutions)**

**Nondeterministic actions**

\[ x : \varphi \text{ means } \text{“choose } x \text{ such that } \varphi \text{”} \]

- Actions can have more than one resolution
- \( \varphi \) is called the before-after-predicate (BAP)
- Variables without tick: before-state
- Variables with tick: after-state.

Example:

\[ x, y : \varphi \quad \text{where} \quad \varphi = x' = y' \land y' > y \]

After the action, \( x \) and \( y \) are equal and \( y \) is strictly greater than before.

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SS 2019 51/96
**Actions** (Generalised Substitutions)

Nondeterministic actions

\[ x :: \varphi \] means “choose \( x \) such that \( \varphi \)”

- Actions can have more than one resolution.
- \( \varphi \) is called the before-after-predicate (BAP).
- Variables without tick: before-state.
- Variables with tick: after-state.

**Example:**

\[ x, y :: x' = y' \land y' > y \]

After the action \( x \) and \( y \) are equal and \( y \) is strictly greater than before the action.
Actions (Generalised Substitutions)

Normal form

Every action can be defined as a before-after-predicate

\[ bap(\text{vars}, \text{vars}', \text{prms}) \]

with

1. \( \text{vars} \) the machine variables before the action
2. \( \text{vars}' \) the machine variables after the action
3. \( \text{prms} \) the parameters of the event

- \( x := t \) is short for \( x :| x' = t \)
- \( x :\in S \) is short for \( x :| x' \in S \)
Initialisation

- Values of the machine in the beginning?
Initialisation

- Values of the machine in the beginning?
- Initial values defined by the special event INITIALISATION.
Initialisation

- Values of the machine in the beginning?
- Initial values defined by the special event INITIALISATION.
- before-after-predicate $bap_{\text{init}}$ and guard $grd_{\text{init}}$ must not refer to $vars$, there is no “before-state”.
Initialisation

- Values of the machine in the beginning?

- Initial values defined by the special event INITIALISATION.

- before-after-predicate $bap_{init}$ and guard $grd_{init}$ must not refer to $vars$, there is no “before-state”.

- After the first state, only normal events trigger.
Machine Semantics

Machine variables \( \text{vars} := v_1, \ldots, v_k \) with types \( \overline{T} = T_1 \times \ldots \times T_k \).

A state \( \sigma \in \overline{T} \) is a vector, variable assignment.

A trace is a sequence of states \( \sigma_0, \sigma_1, \ldots \) such that

- first state \( \sigma_0 \) is result of the initialisation event
- every state \( \sigma_i \) results from an event which operates on \( \sigma_{i-1} \) (for every \( i > 0 \)).

The semantics of a machine \( M \) is the set of all traces possible for \( M \).
Event Parameters

Sources for indeterminism

- indeterministic choices in bap's (cf. $\in, \mid$)
- event parameters

Event parameter may model:

- content of messages passed around
- indeterministic user input
- unpredictable environment actions
- a number, amount of data to operate with
- ...

Technically event parameters can be removed and replaced by existential quantifiers.
Semantics (more formally)

State space: \( \bar{T} = T_1 \times \ldots \times T_k \)
Semantics (more formally)

State space: $\overline{T} = T_1 \times \ldots \times T_k$

Trace: $t \in \mathbb{N} \rightarrow \overline{T}$
Semantics (more formally)

State space: $\overline{T} = T_1 \times \ldots \times T_k$

Trace: $t \in \mathbb{N} \rightarrow \overline{T}$
with

- $\exists \text{prms}_{\text{init}} \cdot \text{grd}_{\text{init}}(\text{prms}_{\text{init}}) \land \text{bap}_{\text{init}}(t(0), \text{prms}_{\text{init}})$
Semantics (more formally)

State space: $\overline{T} = T_1 \times \ldots \times T_k$

Trace: $t \in \mathbb{N} \rightarrow \overline{T}$

with

- $\exists \text{prms}_{init} \cdot \text{grd}_{init}(\text{prms}_{init}) \land \text{bap}_{init}(t(0), \text{prms}_{init})$
- For $n \in \mathbb{N}_1$, there is $e \in \text{EVENTS}$ such that
  $\exists \text{prms}_e \cdot \text{grd}_e(t(i - 1), \text{prms}_e) \land \text{bap}_e(t(i - 1), t(i), \text{prms}_e)$
Semantics (more formally)

State space: $\overline{T} = T_1 \times \ldots \times T_k$

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Semantics (more formally)

State space: \( \overline{T} = T_1 \times \ldots \times T_k \)

Trace: \( t \in \mathbb{N} \rightarrow \overline{T} \)

with

- \( \exists prms_{init} \cdot grd_{init}(prms_{init}) \land bap_{init}(t(0), prms_{init}) \)
- For \( n \in \mathbb{N}_1 \), there is \( e \in EVENTS \) such that
  \( \exists prms_e \cdot grd_e(t(i - 1), prms_e) \land bap_e(t(i - 1), t(i), prms_e) \)

Partial, finite trace trace: \( t \in 0..n \rightarrow \overline{T} \)
Semantics (more formally)

State space: $\overline{T} = T_1 \times \ldots \times T_k$

Trace: $t \in \mathbb{N} \rightarrow \overline{T}$
with

- $\exists \text{prms}_{\text{init}} \cdot \text{grd}_{\text{init}}(\text{prms}_{\text{init}}) \land \text{bap}_{\text{init}}(t(0), \text{prms}_{\text{init}})$
- For $n \in \mathbb{N}_1$, there is $e \in \text{EVENTS}$ such that
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Partial, finite trace trace: $t \in 0..n \rightarrow \overline{T}$

Deadlock: no event $e$ can be triggered, i.e.
$\forall \text{prms}_e \cdot \neg \text{grd}_e(t(n), \text{prms}_e)$ for all events $e$. 
SAFETY: Do all states reachable by $M$ satisfy $inv$?

The red trace violates the invariant in two states.
Proof Obligation INV

To show that \( \text{inv}(\text{vars}) \) is an invariant for machine \( M \), one proves for every event:

- Invariants
- Guards of the event
- Before-after-predicate of the event

\[ \Rightarrow \]

modified invariant
Proof Obligation INV

To show that $inv(\text{vars})$ is an invariant for machine $M$, one proves:

1. $\forall \text{prms}, \text{vars}' \cdot \text{grd}_{\text{init}}(\text{prms}) \land \text{bap}_{\text{init}}(\text{vars}', \text{prms}) \rightarrow inv(\text{vars}')$

   (Invariant initially valid)
Proof Obligation INV

To show that \( \text{inv}(\text{vars}) \) is an invariant for machine \( M \), one proves:

1. \( \forall \text{prms}, \text{vars}'. \quad \text{grd}_{\text{init}}(\text{prms}) \land \text{bap}_{\text{init}}(\text{vars}', \text{prms}) \rightarrow \text{inv}(\text{vars}') \)
   (Invariant initially valid)

2. \( \forall \text{prms}, \text{vars}, \text{vars}'. \quad \text{inv}(\text{vars}) \land \text{grd}_e(\text{vars}, \text{prms}) \land \text{bap}_e(\text{vars}, \text{vars}', \text{prms}) \rightarrow \text{inv}(\text{vars}') \)
   for every event \( e \) in \( M \).
   (Events preserve invariant)
Proof Obligation INV

To show that $inv(vars)$ is an invariant for machine $M$, one proves:

1. $\forall prms, vars'.
   \begin{align*}
   &grd_{init} \land bap_{init} \rightarrow inv \\
   \text{(Invariant initially valid)}
   \end{align*}$

2. $\forall prms, vars, vars'.
   \begin{align*}
   &inv \land grd_{e} \land bap_{e} \rightarrow inv' \\
   \text{for every event } e \text{ in } M. \\
   \text{(Events preserve invariant)}
   \end{align*}$
Proof Obligation INV

To show that $inv(\text{vars})$ is an invariant for machine $M$, one proves:

1. $\forall prms, \text{vars}'.
   \quad grd_{init} \land bap_{init} \rightarrow inv$
   (Invariant initially valid)

2. $\forall prms, \text{vars}, \text{vars}'\cdot
   \quad inv \land grd_e \land
   \quad bap_e \rightarrow inv'$
   for every event $e$ in $M$.
   (Events preserve invariant)

Note: Proof Obligation INV is a sufficient criterion, but not necessary. Necessary for inductive invariants.
Inductive Invariant

MACHINE \textit{IndInv}
VARIABLES $x$
INvariants $x \in \mathbb{Z}$ \hspace{1em} $x \geq 0$
EVENTS

\textbf{INITIALISATION} \triangleq
\hspace{1em} $x := 2$

\textbf{STEP} \triangleq
\hspace{1em} $x := 2 \times (x - 1)$

There is only one trace:

$(2, 2, 2, 2, \ldots)$

invariant is fulfilled.
Inductive Invariant – Won’t prove

Proof obligation INV for event STEP

\[ \text{inv}(x) \land \text{grd}(x) \land \text{bap}(x, x') \rightarrow \text{inv}(x') \]
Inductive Invariant – Won’t prove

Proof obligation INV for event STEP

\[ \text{inv}(x) \land \text{grd}(x) \land \text{bap}(x, x') \rightarrow \text{inv}(x') \]

\[ x \geq 0 \land x' = 2 \times (x - 1) \rightarrow x' \geq 0 \]

This is not valid! Invariant is not inductive.

Counter-example: \( x = 0, \ x' = -2 \)
Inductive Invariant – Won’t prove

Proof obligation INV for event STEP

\[
\begin{align*}
\text{inv}(x) & \land \text{grd}(x) \land \text{bap}(x, x') \rightarrow \text{inv}(x') \\
x \geq 0 & \land x' = 2 \times (x - 1) \rightarrow x' \geq 0
\end{align*}
\]

\[\text{‡} \text{ This is not valid! Invariant is not inductive.} \text{‡}\]

Counter-example: \(x = 0, x' = -2\)
Inductive Invariant – Won’t prove

Proof obligation INV for event STEP

\[
\begin{align*}
\text{inv}(x) & \land \text{grd}(x) \land \text{bap}(x, x') \rightarrow \text{inv}(x') \\
x \geq 0 & \land x' = 2 \ast (x - 1) \rightarrow x' \geq 0
\end{align*}
\]

‡ This is not valid! Invariant is not inductive. ‡
Counter-example: \( x = 0, x' = -2 \)
Show that every action is feasible if the guard is true:

\[
\begin{align*}
\text{Invariants} \\
\text{Guards of the event} \\
\Rightarrow \\
\exists v' \cdot \text{before-after-predicate}
\end{align*}
\]
The action of an event is possible if guard is true.

\[ \forall \text{vars, prms} \cdot \text{grd}_e(\text{vars, prms}) \rightarrow \exists \text{vars'} \cdot \text{bap}(\text{vars, vars', prms}) \]

Deterministic action: \( x := t \)

... nothing to show

Indeterministic action: \( x \in S \)

... show that \( S \neq \emptyset \)

Indeterministic action: \( x :| \varphi \)

... show satisfiability of \( \varphi \)

Thus impossible evolutions like \( x :| \text{false} \) or \( x \in \emptyset \) are avoided
Recap:

**Deadlock**: no event $e$ can be triggered, i.e.

$$\forall prms_e \cdot \neg grd_e(t(n), prms_e)$$

for all events $e$. 
Recap:
Deadlock: no event $e$ can be triggered, i.e.
$\forall prms_e \cdot \neg grd_e(t(n), prms_e)$ for all events $e$.

Proof Obligation
There is always an event that can trigger:

$$\forall vars \cdot inv(vars) \Rightarrow \bigvee_{\text{event } e \in M} \exists prms \cdot grd_e(vars, prms)$$
Recap:
Deadlock: no event \( e \) can be triggered, i.e.
\[
\forall prms_e \cdot \neg grd_e(t(n), prms_e)
\]
for all events \( e \).

Proof Obligation
There is always an event that can trigger:
\[
\forall \text{vars} \cdot \text{inv}(\text{vars}) \Rightarrow \bigvee_{\text{event } e \in M} \exists \text{prms} \cdot \text{grd}_e(\text{vars}, \text{prms})
\]

Again, this is sufficient not necessary.
(The invariant may be too weak to imply deadlock freedom)
Event-B – Refinement
Refinement in Event-B
MACHINE Refinement in Event-B

\[ x \geq 0 \]

EVENTS

\[ \hat{\text{INCREASE}} = x : | x' \geq x \]

MACHINE Refined

REFINES Abstract VARIABLES \[ x \]

EVENTS NEXT

\[ \hat{\text{VARIABLE}} = \text{REFINES} \hat{\text{INCREASE}} \]

\[ x := 5 \times x^2 + 3 \times x \]

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MACHINE Abstract
VARIABLES x
INVARIANTS x ≥ 0
EVENTS INCREASE ≜ 
   x :| x' ≥ x
Refinement in Event-B

MACHINE Abstract
VARIABLES x
INVARIANTS x ≥ 0
EVENTS INCREASE ≜
   x : | x' ≥ x
Refinement in Event-B

MACHINE Abstract
VARIABLES x
INVARIANTS x ≥ 0
EVENTS INCREASE ≜
\[ x : | x' ≥ x \]

MACHINE Refined
REFINES Abstract
VARIABLES x
Refinement in Event-B

MACHINE Abstract
VARIABLES x
INVARIANTS \( x \geq 0 \)
EVENTS INCREASE \( \triangleq \)
\( x :| x' \geq x \)

MACHINE Refined
REFINES Abstract
VARIABLES x
EVENTS nextVal \( \triangleq \)
REFINES INCREASE
\( x := 5 \times x^2 + 3 \times x \)
MACHINE Abstract
VARIABLES x
INVARIANT x ≥ 0
EVENTS INCREASE ≜ 
   x : x' ≥ x

MACHINE Refined
REFINES Abstract
VARIABLES x
EVENTS NEXT VAL ≜ 
   x := 5 * x^2 + 3 * x
Refining Machines

MACHINE Abstract
SEES Context
VARIABLES \( vars_A \)
INVARIANTS
\[ inv_A(vars_A) \]
EVENTS
INITIALISATION \( \cong \) …
EVT\(_A \) \( \cong \) …
END

MACHINE Refined
REFINES Abstract
SEES Context
VARIABLES \( vars_R \)
INVARIANTS
\[ inv_R(vars_A, vars_R) \]
EVENTS
INITIALISATION \( \cong \) …
EVT\(_R \) \( \cong \)
REFINES EVT\(_A \) …
END
Machines as Relations

Every machine $M$ defines:
- a state space $S_M$ spanned by the types of $vars_M$
- the initialisation $I_M \subseteq S_M$
- the transition relations $E_{M;evt} \in S_M \leftrightarrow S_M$ (for event $evt$)

**Details**

$$S_M = \tau(v_1) \times \ldots \times \tau(v_k) \quad \text{(with } vars_M = v_1, \ldots, v_k)$$
Machines as Relations

Every machine $M$ defines:
- a state space $S_M$ spanned by the types of $\text{vars}_M$
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Details

\[
S_M = \tau(v_1) \times \ldots \times \tau(v_k) \quad \text{(with } \text{vars}_M = v_1, \ldots, v_k) \\
I_M(p) = \{ s \in S_M \mid \text{grd}_{\text{init}}(p) \land \text{bap}_{\text{init}}(s', p) \} 
\]
Machines as Relations

Every machine $M$ defines:

- a state space $S_M$ spanned by the types of $\text{vars}_M$
- the initialisation $I_M \subseteq S_M$
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Details

$$S_M = \tau(v_1) \times \ldots \times \tau(v_k) \quad \text{(with vars}_M = v_1, \ldots, v_k)$$

$$I_M(p) = \{ s \in S_M \mid \text{grd}_{\text{init}}(p) \land \text{bap}_{\text{init}}(s', p) \}$$

$$I_M = \bigcup_{p} I_M(p)$$
Machines as Relations

Every machine $M$ defines:
- a state space $S_M$ spanned by the types of $\text{vars}_M$
- the initialisation $I_M \subseteq S_M$
- the transition relations $E_{M;\text{evt}} \in S_M \leftrightarrow S_M$ (for event $\text{evt}$)

Details

<table>
<thead>
<tr>
<th>$S_M$</th>
<th>$\tau(v_1) \times \ldots \times \tau(v_k)$ (with $\text{vars}_M = v_1, \ldots, v_k$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_M(p)$</td>
<td>${s \in S_M \mid \text{grd}<em>{\text{init}}(p) \land \text{bap}</em>{\text{init}}(s', p)}$</td>
</tr>
<tr>
<td>$I_M$</td>
<td>$\bigcup_{p} I_M(p)$</td>
</tr>
<tr>
<td>$E_{M;\text{evt}}(p)$</td>
<td>${(s \mapsto s') \mid \text{grd}<em>{\text{evt}}(s, p) \land \text{bap}</em>{\text{evt}}(s, s', p)}$</td>
</tr>
</tbody>
</table>
Machines as Relations

Every machine $M$ defines:

- a state space $S_M$ spanned by the types of $\text{vars}_M$
- the initialisation $I_M \subseteq S_M$
- the transition relations $E_{M;\text{evt}} \in S_M \leftrightarrow S_M$ (for event $\text{evt}$)

Details

\[
S_M = \tau(v_1) \times \ldots \times \tau(v_k) \quad \text{(with vars}_M = v_1, \ldots, v_k)
\]

\[
I_M(p) = \{ s \in S_M \mid \text{grd}_{\text{init}}(p) \wedge \text{bap}_{\text{init}}(s', p) \}
\]

\[
I_M = \bigcup_p I_M(p)
\]

\[
E_{M;\text{evt}}(p) = \{ (s \mapsto s') \mid \text{grd}_{\text{evt}}(s, p) \wedge \text{bap}_{\text{evt}}(s, s', p) \}
\]

\[
E_M;\text{evt} = \bigcup_p E_{M;\text{evt}}(p)
\]
Simple Refinement – Definition

Every trace of the refined machine $R$ is a trace of the abstract machine $A$.

**Definition: Simple Refinement**

Let $R, A$ be two machines with the same state space $S$. $R$ is called a refinement of $A$ if

1. $I_R \subseteq I_A$ and
Every trace of the refined machine $R$ is a trace of the abstract machine $A$.

**Definition: Simple Refinement**

Let $R$, $A$ be two machines with the same state space $S$. $R$ is called a refinement of $A$ if

1. $I_R \subseteq I_A$ and 
2. $E_{R;\text{evt}_R} \subseteq E_{A;\text{evt}_A}$ for each event
Every trace of the refined machine $R$ is a trace of the abstract machine $A$.

**Definition: Simple Refinement**

Let $R, A$ be two machines with the same state space $S$. $R$ is called a refinement of $A$ if

1. $I_R \subseteq I_A$ and
2. $E_{R;evt_R} \subseteq E_{A;evt_A}$ for each event

$(evt_R$ is the event in $R$ that refines event $evt_A$ from $A)$
Loss of behaviour

Why is this problematic?

MACHINE A
EVENT emergencyStop ≜ WHERE true THEN heavyMachine := stop END

refined by

MACHINE R
EVENT emergencyStop ≜ EVENT emergencyStop WHERE false THEN heavyMachine := stop END
Loss of behaviour

Why is this problematic?

MACHINE A ... 
EVENT emergencyStop ≡
WHERE true THEN heavyMachine := stop 
END

refined by

MACHINE R ... 
EVENT emergencyStop ≡ REFINES emergencyStop
WHERE false THEN heavyMachine := stop 
END
Loss of behaviour

Why is this problematic?

MACHINE $A$ ... 
EVENT $\text{emergencyStop} \triangleq$ 
WHERE $true$ THEN $\text{heavyMachine} := \text{stop}$ 
END

refined by

MACHINE $R$ ... 
EVENT $\text{emergencyStop} \triangleq \text{REFINES} \text{emergencyStop}$ 
WHERE $false$ THEN $\text{heavyMachine} := \text{stop}$ 
END

$E_{R; \text{evt}} = \emptyset \implies R \text{ refines } A$
Loss of behaviour

Every trace for $A$ has a refining trace for $R$. 
Loss of behaviour

Every trace for $A$ has a refining trace for $R$.

More precisely

For every trace in $A$ with triggered events $evt_{A,1}, evt_{A,2}, \ldots$, there is a trace in $R$ with triggered events $evt_{R,1}, evt_{R,2}, \ldots$ and $evt_{R;i}$ refines $evt_{A;i}$. 
Loss of behaviour

Every trace for $A$ has a refining trace for $R$.

More precisely

For every trace in $A$ with triggered events $evt_{A,1}, evt_{A,2}, \ldots$, there is a trace in $R$ with triggered events $evt_{R,1}, evt_{R,2}, \ldots$ and $evt_{R;i}$ refines $evt_{A;i}$.

Definition: Lockfree Refinement

Let $R, A$ be two machines with the same state space $S$. $R$ is called a lockfree refinement of $A$ if

1. $I_R \subseteq I_A$
Loss of behaviour

Every trace for $A$ has a refining trace for $R$.

More precisely

For every trace in $A$ with triggered events $\text{evt}_{A,1}, \text{evt}_{A,2}, \ldots$, there is a trace in $R$ with triggered events $\text{evt}_{R,1}, \text{evt}_{R,2}, \ldots$ and $\text{evt}_{R;i}$ refines $\text{evt}_{A;i}$.

Definition: Lockfree Refinement

Let $R, A$ be two machines with the same state space $S$. $R$ is called a lockfree refinement of $A$ if

1. $I_R \subseteq I_A$
2. $I_R \neq \emptyset$
Loss of behaviour

Every trace for $A$ has a refining trace for $R$.

More precisely

For every trace in $A$ with triggered events $\text{evt}_A, 1, \text{evt}_A, 2, \ldots$, there is a trace in $R$ with triggered events $\text{evt}_R, 1, \text{evt}_R, 2, \ldots$ and $\text{evt}_R; i$ refines $\text{evt}_A; i$.

Definition: Lockfree Refinement

Let $R, A$ be two machines with the same state space $S$. $R$ is called a lockfree refinement of $A$ if

1. $I_R \subseteq I_A$
2. $I_R \neq \emptyset$
3. $E_R; \text{evt}_R \subseteq E_A; \text{evt}_A$ for each event
Loss of behaviour

Every trace for $A$ has a refining trace for $R$.

More precisely

For every trace in $A$ with triggered events $evt_{A,1}$, $evt_{A,2}$, $\ldots$, there is a trace in $R$ with triggered events $evt_{R,1}$, $evt_{R,2}$, $\ldots$ and $evt_{R,i}$ refines $evt_{A,i}$.

Definition: Lockfree Refinement

Let $R$, $A$ be two machines with the same state space $S$. $R$ is called a lockfree refinement of $A$ if

1. $I_R \subseteq I_A$
2. $I_R \neq \emptyset$
3. $E_{R;evt_R} \subseteq E_{A;evt_A}$ for each event
4. $\text{dom}(E_{A;evt_A}) \subseteq \text{dom}(E_{R;evt_R})$ for each event
Coupling

More general notion of refinement

What if abstract machine $A$ and refinement $R$ have different state spaces $S_A$ and $S_R$?
Coupling

More general notion of refinement

What if abstract machine $A$ and refinement $R$ have different state spaces $S_A$ and $S_R$?

$\implies \textbf{Couple}$ abstract and refined state space.

$$C \in S_R \leftrightarrow S_A$$

\textbf{Coupling invariant / Gluing invariant}
Coupling

More general notion of refinement

What if abstract machine $A$ and refinement $R$ have different state spaces $S_A$ and $S_R$?

→ **Couple** abstract and refined state space.

$$C \in S_R \leftrightarrow S_A$$

**Coupling invariant / Gluing invariant**

---

**Example**

- **MACHINE** AbstractFileSys
  **VARIABLES** openFiles
  **INVARIANTS**
  $$\text{openFiles} \subseteq \text{FILES}$$

- **MACHINE** RefinedFileSys
  **VARIABLES** openModes
  **INVARIANTS**
  $$\text{openModes} \subseteq \text{FILES} \times \text{MODES}$$
Coupling

More general notion of refinement

What if abstract machine $A$ and refinement $R$ have different state spaces $S_A$ and $S_R$?

→ **Couple** abstract and refined state space.

$C \in S_R \leftrightarrow S_A$  
**Coupling invariant / Gluing invariant**

Example

<table>
<thead>
<tr>
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<tr>
<td>VARIABLES</td>
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<tr>
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<td>openModes</td>
</tr>
<tr>
<td>INVARIANTS</td>
<td>$openModes \subseteq FILES \times MODES$</td>
</tr>
</tbody>
</table>

$C = \{ r \mapsto a \mid a = \text{dom}(r) \} = \{ f, m \cdot (f \mapsto m) \mapsto m \}$
Sensible to assume $C$ a total relation:

$$C \in S_R \leftrightarrow S_A$$

Often, coupling is a total function:

$$C \in S_R \rightarrow S_A$$

Define *one* abstraction for any detailed state. BUT sometimes, several possible abstractions per concrete state sensible.
Refinement – Coupled Traces

For every concrete trace \((\chi_0, \chi_1, \ldots)\) of \(R\) with events \((\text{evt}_R^1, \text{evt}_R^2, \ldots)\) there exists an abstract trace \((\sigma_0, \sigma_1, \ldots)\) with events \((\text{evt}_A^1, \text{evt}_A^2, \ldots)\) such that

\[ \chi_i \mapsto \sigma_i \in C \text{ for all } i \in \mathbb{N} \]

\text{evt}_R^i \text{ refines } \text{evt}_A^i.
Refinement – Coupled Traces

For every concrete trace \((\chi_0, \chi_1, \ldots)\) of \(R\) with events \((\text{evt}_R^1, \text{evt}_R^2, \ldots)\) there exists an abstract trace \((\sigma_0, \sigma_1, \ldots)\) with events \((\text{evt}_A^1, \text{evt}_A^2, \ldots)\) such that \(\chi_i \mapsto \sigma_i \in C\) for all \(i \in \mathbb{N}\).
Refinement: $R$ refines $A$

For every concrete trace $(\chi_0, \chi_1, \ldots)$ of $R$ with events $(\text{evt}_1^R, \text{evt}_2^R, \ldots)$ there exists an abstract trace $(\sigma_0, \sigma_1, \ldots)$ with events $(\text{evt}_1^A, \text{evt}_2^A, \ldots)$ such that

$\chi_i \mapsto \sigma_i \in C$ for all $i \in \mathbb{N}$
Refinement – Coupled Traces

Refinement: $R$ refines $A$

For every concrete trace $(\chi_0, \chi_1, \ldots)$ of $R$ with events $(\text{evt}_1^R, \text{evt}_2^R, \ldots)$ there exists an abstract trace $(\sigma_0, \sigma_1, \ldots)$ with events $(\text{evt}_1^A, \text{evt}_2^A, \ldots)$ such that

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For every concrete trace \((\chi_0, \chi_1, \ldots)\) of \(R\) with events \((\text{evt}_1^R, \text{evt}_2^R, \ldots)\) there exists an abstract trace \((\sigma_0, \sigma_1, \ldots)\) with events \((\text{evt}_1^A, \text{evt}_2^A, \ldots)\) such that

1. \(\chi_i \mapsto \sigma_i \in C\) for all \(i \in \mathbb{N}\)
2. \(\text{evt}_i^R\) refines event \(\text{evt}_i^A\).
Definition: Refinement

Let $R, A$ be two machines with state spaces $S_R, S_A$. Let $C \in S_R \leftrightarrow R_A$ be the coupling invariant. $R$ is called a refinement of $A$ modulo $C$ if

1. $I_R \subseteq C^{-1}[I_A]$
2. $E_R; \text{evt} R \subseteq C; E_A; \text{evt} A; C^{-1}$ for each event.
Refinement – Definition

Definition: Refinement

Let $R$, $A$ be two machines with state spaces $S_R$, $S_A$. Let $C \in S_R \leftrightarrow R_A$ be the coupling invariant. $R$ is called a refinement of $A$ modulo $C$ if

1. $I_R \subseteq C^{-1}[I_A]$ and

$$
(\forall x, y \cdot x \leftrightarrow y \in R^{-1} \iff y \leftrightarrow x \in R, \text{ inverse relation})
$$
Refinement – Definition

Definition: Refinement

Let $R, A$ be two machines with state spaces $S_R, S_A$. Let $C \in S_R \leftrightarrow R_A$ be the coupling invariant. $R$ is called a refinement of $A$ modulo $C$ if

1. $I_R \subseteq C^{-1}[I_A]$ and
2. $E_{R;evt_R} \subseteq C \circ E_{A;evt_A} \circ C^{-1}$ for each event.

$$(\forall x, y : x \mapsto y \in R^{-1} \iff y \mapsto x \in R, \text{ inverse relation})$$
Refinement – Path subsumption

\[ \sigma_n \xrightarrow{\text{evt}_A} \sigma_{n+1} \]

\[ \chi_n \xleftarrow{\text{evt}_R} \chi_{n+1} \]

\[ C \subseteq E_R; \]

\[ \text{evt}_R \subseteq C; \]

\[ E_A; \]

\[ C - 1 \]
Refinement – Path subsumption

\begin{align*}
\sigma_n &\xrightarrow{\text{evt}_A} \sigma_{n+1} \\
\chi_n &\xleftarrow{\text{evt}_R} \chi_{n+1}
\end{align*}

\[ E_R;\text{evt}_R \subseteq C ; E_A;\text{evt}_A ; C^{-1} \]
Refinement – Path subsumption

\[ \sigma_n \xrightarrow{\text{evt}_A} \sigma_{n+1} \]

\[ C \xleftarrow{\chi_n} C \]

\[ \chi_n \xrightarrow{\text{evt}_R} \chi_{n+1} \]

\[ E_R;\text{evt}_R \subseteq C ; E_A;\text{evt}_A ; C^{-1} \]
Refinement – Path subsumption

\[ \sigma_n \xrightarrow{\text{evt}_A} \sigma_{n+1} \]

\[ C \xrightarrow{\text{evt}_R} C \]

\[ E_{R;\text{evt}_R} \subseteq C ; E_{A;\text{evt}_A} ; C^{-1} \]
Refinement – Path subsumption

\[ \sigma_n \xrightarrow{\text{evt}_A} \sigma_{n+1} \]

\[ C \subseteq E_R \]

\[ \chi_n \xrightarrow{\text{evt}_R} \chi_{n+1} \]

\[ E_R;\text{evt}_R \subseteq C \wedge E_A;\text{evt}_A \wedge C^{-1} \]
The coupling invariant is specified as part of the invariant of the refining machine.
Specifying Coupling

The coupling invariant is specified as part of the invariant of the refining machine.

The invariant of a refinement is allowed to refer to variables of its abstraction.
The coupling invariant is specified as part of the invariant of the refining machine.

The invariant of a refinement is allowed to refer to variables of its abstraction.

Example (from slide 72)

<table>
<thead>
<tr>
<th>MACHINE AbstractFileSys</th>
</tr>
</thead>
<tbody>
<tr>
<td>VARIABLES openFiles</td>
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<tr>
<td>INVARIANTS openFiles ⊆ FILES</td>
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<table>
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<td>INVARIANTS openModes ⊆ FILES × MODES</td>
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Specifying Coupling

The coupling invariant is specified as part of the invariant of the refining machine.

The invariant of a refinement is allowed to refer to variables of its abstraction.

Example (from slide 72)

MACHINE AbstractFileSys
VARIABLES openFiles
INVARIANTS
  openFiles ⊆ FILES

MACHINE RefinedFileSys
VARIABLES openModes
INVARIANTS
  openModes ⊆ FILES × MODES
  openFiles = dom(openModes)
Proof Obligation GRD

Proof that event guard in refinement is stronger than in abstract machine.

⇒ Abstraction is enabled when refinement is.

Abstract invariants
Concrete invariants
Concrete event guard

⇒

Abstract event guard
Proof Obligation GRD

Proof that event guard in refinement is stronger than in abstract machine.

⇒ Abstraction is enabled when refinement is.

Abstract invariants
Concrete invariants
Concrete event guard
⇒
Abstract event guard

∀vars_A, vars_R ·

inv_A(vars_A) ∧ inv_R(vars_A, vars_R) ∧ grd_R(vars_R)
⇒ grd_A(vars_A)

(Version w/o parameters, see literature for full version)
Proof Obligation SIM

Show that refined action *simulates* abstract actions

Abstract invariants
Concrete invariants
Concrete event guard
Concrete before-after-predicate

⇒
Abstract before-after-predicate
Proof Obligation SIM

Show that refined action *simulates* abstract actions

- Abstract invariants
- Concrete invariants
- Concrete event guard
- Concrete before-after-predicate

\[ \Rightarrow \]
- Abstract before-after-predicate

**Rem** \( E_{R;evt_R} \subseteq C \; ; \; E_{A;evt_A} \; ; \; C^{-1} \)
Proof Obligation SIM

Show that refined action *simulates* abstract actions

Abstract invariants
Concrete invariants
Concrete event guard
Concrete before-after-predicate

\[ \Rightarrow \]
Abstract before-after-predicate

**Rem** \( E_{R;evt_R} \subseteq C ; E_{A;evt_A} \ ; C^{-1} \)

**Obs** The coupling invariant is only used for the before-state not for the after-state.
Proof Obligation **SIM**

Show that refined action *simulates* abstract actions

Abstract invariants
Concrete invariants
Concrete event guard
Concrete before-after-predicate

\[ \implies \]
Abstract before-after-predicate

**Rem** \( E_{R;\text{evt}_R} \subseteq C ; E_{A;\text{evt}_A} ; C^{-1} \)

**Obs** The coupling invariant is only used for the before-state not for the after-state.

? Why?
Show that refined action simulates abstract actions

Abstract invariants
Concrete invariants
Concrete event guard
Concrete before-after-predicate

⇒

Abstract before-after-predicate

Rem \( E_{R;evt_R} \subseteq C \; E_{A;evt_A} \; C^{-1} \)

Obs The coupling invariant is only used for the before-state not for the after-state.

? Why?

! Already proven condition INV implies invariant for after-state.
Event-B has more ...

Things not covered in these slides:

- Witnesses for parameters dropped in refinements
- Termination issues (variants)
- Extended/Not extended events
- Event merging
- Sequential refinement
- ...

Event-B has more …

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Beckert, Ulbrich – Applications of Formal Verification
Event-B has more ...

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Beckert, Ulbrich – Applications of Formal Verification
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...
Event-B has more ...

Things not covered in these slides:

- Witnesses for parameters dropped in refinements
- Termination issues (variants)
- Extended/Not extended events
- Event merging
- Sequential refinement
- ..
Byzantine Agreement –
A case study verified with Event-B

Based on:
Byzantine Generals

“When shall we attack?”

agree even in the presence of traitors
Byzantine Generals

“When shall we attack?”
Byzantine Generals

“When shall we attack?”
Byzantine Generals

“When shall we attack?”

agree on a time even in the presence of traitors
Application in Avionics

Which components are operative?

C1, C2, C3, C4 agree on the set of operative components even in the presence of faulty components.
Application in Avionics

“Which components are operative?”

C1
C2
C3
C4
Application in Avionics

“Which components are operative?”

C1

C2

C3

C4

agree on the set of operative components even in the presence of faulty components
Explanation by Example

C1

C2

C3

C4

CONSENSUS!
Explanation by Example
Explanation by Example
Explanation by Example

C1

C2

C3

C4

CONSENSUS!
Explanation by Example

C1

C2

C3

C4

CONSENSUS!
Explanation by Example

C1

C2

C3

C4

CONSENSUS!
Explanation by Example
Explanation by Example

C1

C2

C3

C4

1,3,2 → 3
2 → 2
3 → 3

2,1,3

3,1,2
Explanation by Example

CONSENSUS!

C1
1,3,2

C2

C3
2,1,3

C4
3,1,2
Example Run 2

C1
C2
C3
C4
Example Run 2

Round 0

C1

C2

C3

C4

Beckert, Ulbrich – Applications of Formal Verification
Example Run 2

Round 1

C2 — C4

C1

C3

Round 0

Beckert, Ulbrich – Applications of Formal Verification
Example Run 2

Round 0

Round 1

Round 2
Byzantine Agreement Algorithm

Verification Goals:

**Validity**  If the transmitter $tt$ is non-faulty, then all non-faulty receivers agree on the value sent by $tt$.

**Agreement**  Any two non-faulty receivers agree on the value assigned to $tt$. 
Byzantine Agreement Algorithm

Round 0: Transmitter sends signed message to all receivers.
Byzantine Agreement Algorithm

Round 0: Transmitter sends signed message to all receivers.

Round $n$: Component receive messages, verify signatures, sign messages and pass them on.
Byzantine Agreement Algorithm

Round 0: Transmitter sends signed message to all receivers.

Round $n$: Component receive messages, verify signatures, sign messages and pass them on.

GOAL: Prove that this algorithm has the “validity” and “agreement” properties.
We know of no area in computer science or mathematics in which informal reasoning is more likely to lead to errors than in the study of this type of algorithm.

Taken from: *The Byzantine Generals Problem*

Leslie Lamport, Robert Shostak, and Marshall Pease
ACM Transactions on Programming Languages and Systems
## Context for Byzantine Agreement

<table>
<thead>
<tr>
<th>CONTEXT</th>
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</thead>
<tbody>
<tr>
<td>SETS</td>
<td></td>
</tr>
<tr>
<td>CONSTANTS</td>
<td></td>
</tr>
<tr>
<td>AXIOMS</td>
<td></td>
</tr>
</tbody>
</table>

END
CONTEXT \textit{Context}

SETS
\begin{itemize}
\item MODULE
\item VALUE
\end{itemize}

CONSTANTS

AXIOMS

END
CONTEXT Context
SETS
   MODULE
   VALUE
CONSTANTS
   faulty, transmitter, V₀
AXIOMS

END
CONTEXT Context

SETS

MODULE
VALUE

CONSTANTS

faulty, transmitter, V₀

AXIOMS

faulty ⊆ MODULE
transmitter ∈ MODULE
V₀ ∈ VALUE
finite(faulty)

END
First machine

MACHINE Messages
SEES Context
VARIABLES
INvariants

...
MACHINE Messages
SEES Context
VARIABLES messages, round, collected
INVARINTS

...
MACHINE Messages
SEES Context

VARIABLES messages, round, collected

INVARIANTS

\[ \text{ty\_mess} : \text{messages} \subseteq \text{MODULE} \times \text{MODULE} \times \text{VALUE} \]

\[ \ldots \]

\[ \text{messages} \] messages being sent in the current round
MACHINE Messages
SEES Context

VARIABLES messages, round, collected

INVARIANTS

ty_mess : messages ⊆ Module × Module × Value
ty_round : round ∈ \mathbb{N}

...
First machine

MACHINE Messages
SEES Context

VARIABLES messages, round, collected

IN VariantS

\[ \text{ty\_mess} : \text{messages} \subseteq \text{MODULE} \times \text{MODULE} \times \text{Value} \]
\[ \text{ty\_round} : \text{round} \in \mathbb{N} \]
\[ \text{ty\_collected} : \text{collected} \in \text{MODULE} \rightarrow \mathcal{P}(\text{Value}) \]

...
First machine (2)

messages messages being sent in the current round

round the number of the current round

collected values observed in previous rounds
First machine (2)

messages messages being sent in the current round

round the number of the current round

collected values observed in previous rounds

MACHINE Messages SEES Context
VARIABLES messages, round, collected
INVARIANTS...
EVENTS

Initialisation \(\triangleq \ldots\)

EVENT ROUND \(\triangleq \)
  act1 : round := round + 1
  act2 : messages :\(\in\ P(Module \times Module \times Value)\)
  act3 : collected := \(\lambda m \cdot collected(m) \cup\)
END
First machine (2)

messages messages being sent in the current round
round the number of the current round
collected values observed in previous rounds

MACHINE Messages SEES Context
VARIABLES messages, round, collected
INVARIANTS...
EVENTS

Initialisation \equiv \ldots

EVENT ROUND \equiv
act1 : \text{round} := \text{round} + 1
act2 : \text{messages} \in \mathcal{P}(\text{MODULE} \times \text{MODULE} \times \text{VALUE})
act3 : \text{collected} := \lambda m \cdot \text{collected}(m) \cup
END
First machine (2)

messages  messages being sent in the current round
round     the number of the current round
collected values observed in previous rounds

MACHINE Messages SEES Context
VARIABLES messages, round, collected
INVARIANTS...
EVENTS
  Initialisation ≜ ...
  EVENT ROUND ≜
    act1 : round := round + 1
    act2 : messages ∈ IP(Module × Module × Value)
    act3 : collected := λm · collected(m) ∪
END
First machine (2)

messages messages being sent in the current round
round the number of the current round
collected values observed in previous rounds

MACHINE Messages SEES Context
VARIABLES messages, round, collected
INVARINTS...
EVENTS

Initialisation \( \equiv \ldots \)

EVENT ROUND \( \equiv \)

act1 : round := round + 1
act2 : messages :\( \in \mathbb{P}(\text{MODULE} \setminus \{\text{transmitter}\} \times \text{MODULE} \times \text{VALUE}) \)
act3 : collected := \( \lambda m \cdot \text{collected}(m) \cup \)

END
First machine (2)

messages messages being sent in the current round
round the number of the current round
collected values observed in previous rounds

MACHINE Messages SEES Context
VARIABLES messages, round, collected
INVARIANTS...
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Initialisation ≜ ...

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act1 : round := round + 1
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MACHINE Messages SEES Context
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EVENTS

Initialisation ≡ ...

EVENT ROUND ≡
  act1 : round := round + 1
  act2 : messages ∈ IP(MODULE \ \{transmitter\} × MODULE × VALUE)
  act3 : collected := λm · collected(m) ∪ \{v | (s, m, v) ∈ messages\}
END
All messages are signed in a trustworthy manner:

No forgery possible $\implies$ Consider only \textit{relayed} messages.
First refinement: signed messages

All messages are signed in a trustworthy manner:

No forgery possible $\implies$ Consider only \textit{relayed} messages.

round $k$:

\[
\begin{align*}
s & \rightarrow v \\
v & \rightarrow r
\end{align*}
\]
First refinement: signed messages

All messages are signed in a trustworthy manner:

No forgery possible \( \implies \) Consider only \textit{relayed} messages.

round \( k \):

\[
\text{s} \xrightarrow{\vee} \text{r}
\]

round \( k + 1 \):

\[
\text{r} \xrightarrow{\vee} \text{n}
\]
Signed messages (2)

round \( k \): \( s \xrightarrow{\nu} r \)

round \( k + 1 \): \( r \xrightarrow{\nu} n \)

MACHINE SignedMessages REFINES Messages

VARIABLES messages, round, collected

INVARINTS

val1: \( \forall s, r, v \cdot (s, r, v) \in \text{messages} \Rightarrow v \in \text{collected(transmitter)} \)

val2: \( \forall n \cdot \text{collected}(n) \subseteq \text{collected(transmitter)} \)

EVENTS

END
Signed messages (2)

round $k$: $s \rightarrow^v r$

round $k + 1$: $r \rightarrow^v n$

MACHINE SignedMessages  REFINES Messages

VARIABLES messages, round, collected

INVARIANTS

val1: $\forall s, r, v \cdot (s, r, v) \in \text{messages} \Rightarrow v \in \text{collected(} \text{transmitter} \text{)}$

val2: $\forall n \cdot \text{collected}(n) \subseteq \text{collected}(\text{transmitter})$

EVENTS

EVENT ROUND REFINES ROUND $\equiv$

act1, act3 as above

act2: $\text{messages} : \in \mathbb{P} (\{(r, n, v) \mid (s, r, v) \in \text{messages}\})$

END
Signed messages (2)

MACHINE SignedMessages REFINES Messages

VARIABLES messages, round, collected

INvariants
val1: ∀s, r, v · (s, r, v) ∈ messages ⇒ v ∈ collected(transmitter)
val2: ∀n · collected(n) ⊆ collected(transmitter)

EVENTS

EVENT ROUND REFINES ROUND ≡
act1, act3 as above
act2: messages ∈ \( \mathcal{P} \{ (r, n, v) | (s, r, v) ∈ messages \} \)
was : messages ∈ \( \mathcal{P}(\text{MODULE \setminus \{transmitter\}} \times \text{MODULE} \times \text{VALUE}) \)

END
Signed messages (2)

\[
\begin{align*}
\text{round } k: \quad & s \xrightarrow{v} r \\
\text{round } k + 1: \quad & r \xrightarrow{v} n
\end{align*}
\]

MACHINE SignedMessages REFINES Messages

VARIABLES messages, round, collected

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EVENTS

EVENT ROUND REFINES ROUND =
act1, act3 as above
act2: messages :\( \in \mathcal{P} \left( \{ (r, n, v) \mid (s, r, v) \in \text{messages} \} \right) \)
was : messages :\( \in \mathcal{P}(\text{MODULE} \setminus \{ \text{transmitter} \} \times \text{MODULE} \times \text{VALUE}) \)

END
Signed messages (2)

\[ \begin{align*}
\text{round } k: & \quad s \overset{v}{\rightarrow} r \\
\text{round } k + 1: & \quad r \overset{v}{\rightarrow} n
\end{align*} \]

MACHINE *SignedMessages* \text{ REFINES } *Messages*

VARIABLES messages, round, collected

INvariants
\begin{align*}
\text{val1: } & \forall s, r, v \cdot (s, r, v) \in \text{messages} \Rightarrow v \in \text{collected}(\text{transmitter}) \\
\text{val2: } & \forall n \cdot \text{collected}(n) \subseteq \text{collected}(\text{transmitter})
\end{align*}

EVENTS

EVENT *ROUND* \text{ REFINES } *ROUND* \equiv
\begin{align*}
\text{act1, act3 } & \text{ as above} \\
\text{act2: } & \text{messages} \in P \left( \{(r, n, v) \mid (s, r, v) \in \text{messages}\} \right) \\
\text{was: } & \text{messages} \in P(\text{MODULE} \setminus \{\text{transmitter}\} \times \text{MODULE} \times \text{VALUE})
\end{align*}

END
Signed messages (2)

round $k$: $s \xrightarrow{v} r$

round $k + 1$: $r \xrightarrow{v} n$

MACHINE SignedMessages REFINES Messages

VARIABLES messages, round, collected

INVARIANTS
  val1: $\forall s, r, v \cdot (s, r, v) \in \text{messages} \Rightarrow v \in \text{collected}(\text{transmitter})$
  val2: $\forall n \cdot \text{collected}(n) \subseteq \text{collected}(\text{transmitter})$

EVENTS

EVENT ROUND REFINES ROUND $\triangleq$
  act1, act3 as above
  act2: $\text{messages} \in \mathcal{P} \left( \{(r, n, v) | (s, r, v) \in \text{messages}\} \right)$
  was : $\text{messages} \in \mathcal{P} (\text{MODULE} \setminus \{\text{transmitter}\} \times \text{MODULE} \times \text{VALUE})$

END
Refinement Tower

Changes message representation:

\[ \text{msgs} \subseteq \text{Module} \times \text{Module} \times \mathcal{P} (\text{Module}) \times \text{Value} \]
non-faulty modules behave well:

\[ r \notin \text{faulty} \land (s, r, h, v) \in \text{msgs} \implies \]
\[ \forall n \cdot (n \notin h \implies (r, n, h \cup \{r\}, v) \in \text{msgs}') \]
hybrid fault model:

\[ \text{faulty} = \text{arbFault} \cup \text{symFaulty} \]

\[ \text{arbFaulty} \cap \text{symFaulty} = \emptyset \]
new event structure:

    PROCESS_EVENT refines SKIP

modifies internal data structures (invisible to abstract machine) and

    ROUND_SWITCH refines ROUND

reproduces the effect of a round change from the internal data.

**An implementation would refine PROCESS_EVENT.**
Agreement!

In machine Guarantees:

\[ \text{round} \geq \text{card}(\text{faulty}) + 1 \implies \]
\[ \left( \forall n, m \cdot n \notin \text{faulty} \land m \notin \text{faulty} \implies \right. \]
\[ \text{collected}(n) = \text{collected}(m) \]
Agreement!

In machine Guarantees:

\[ \text{round} \geq \text{card}(\text{faulty}) + 1 \implies \]
\[ (\forall n, m \cdot n \notin \text{faulty} \land m \notin \text{faulty} \implies \]
\[ \text{collected}(n) = \text{collected}(m) ) \]

In machine HybridGuarantees:

\[ \text{round} \geq \text{card}(\text{arbFaulty}) + 1 \implies \]
\[ (\forall n, m \cdot n \notin \text{faulty} \land m \notin \text{faulty} \implies \]
\[ \text{collected}(n) = \text{collected}(m) ) \]
**Verification Effort**

<table>
<thead>
<tr>
<th>Numbers</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size:</td>
<td>4 contexts, 12 machines, 106 invariants</td>
</tr>
<tr>
<td>Labour:</td>
<td>approx. 4 person months</td>
</tr>
<tr>
<td>Proofs:</td>
<td>322 proof obligations</td>
</tr>
<tr>
<td>Automation:</td>
<td>74/322, 23%</td>
</tr>
</tbody>
</table>