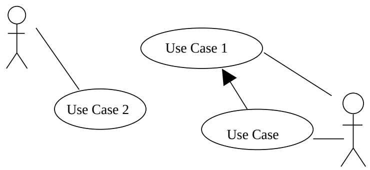
Entwicklung objektorientierter Software mit formalen Methoden

Program Verification – Dynamic Logic for Users

Bernhard Beckert

UNIVERSITÄT KOBLENZ-LANDAU

Verification in different design phases



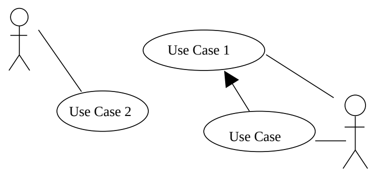
Analyse
Diagrams

+

Requirements
OCL + nat. Language

time

Verification in different design phases



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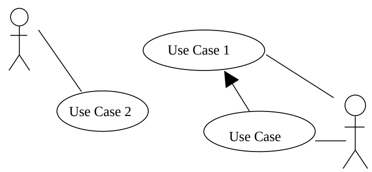

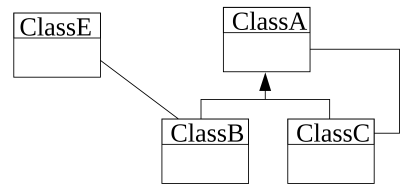

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Specification

OCL (inv., pre−/post)

time

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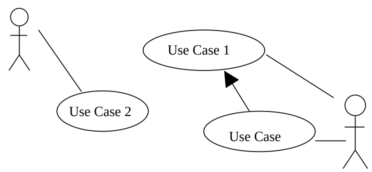

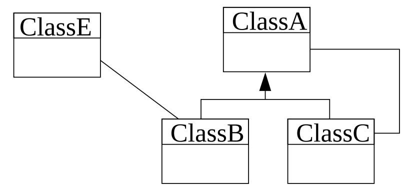

Analyse Diagrams	Design Diagrams
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Refinement

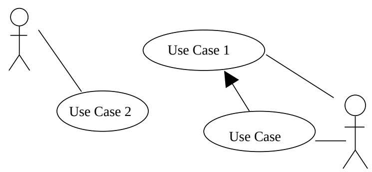
time

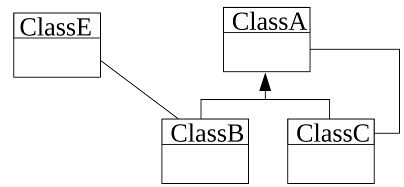
—

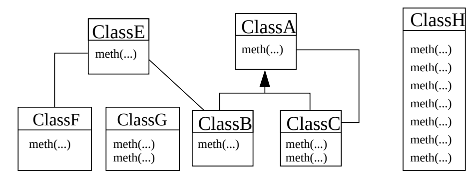
(semantic gap)

Horizontal Verification

Verification in different design phases







Analyse Diagrams	Design Diagrams	Implementation Diagrams
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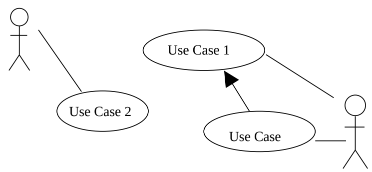

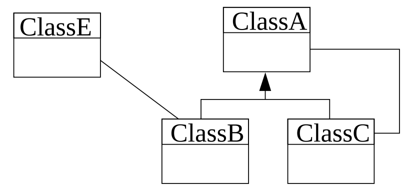

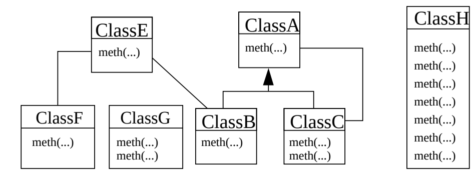

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Refinement

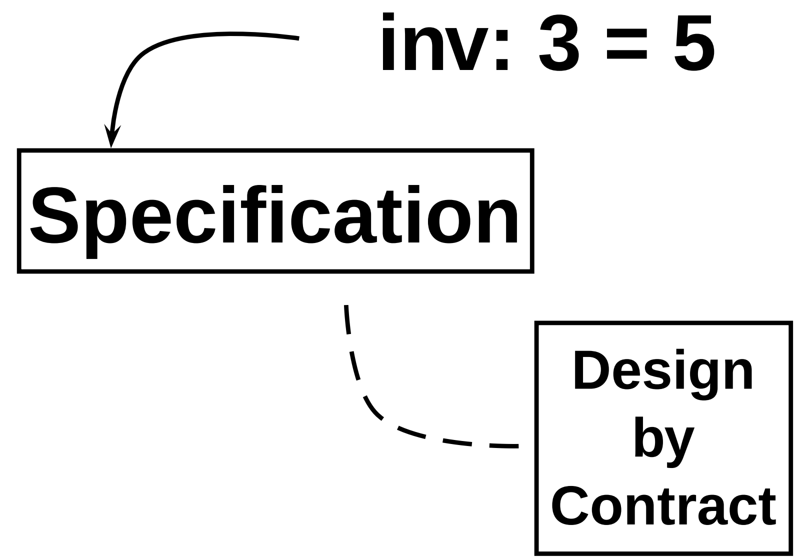
Equivalence

time →

— (semantic gap)	Horizontal Verification	Vertical Verification

What has to be proved?

Horizontal Verification



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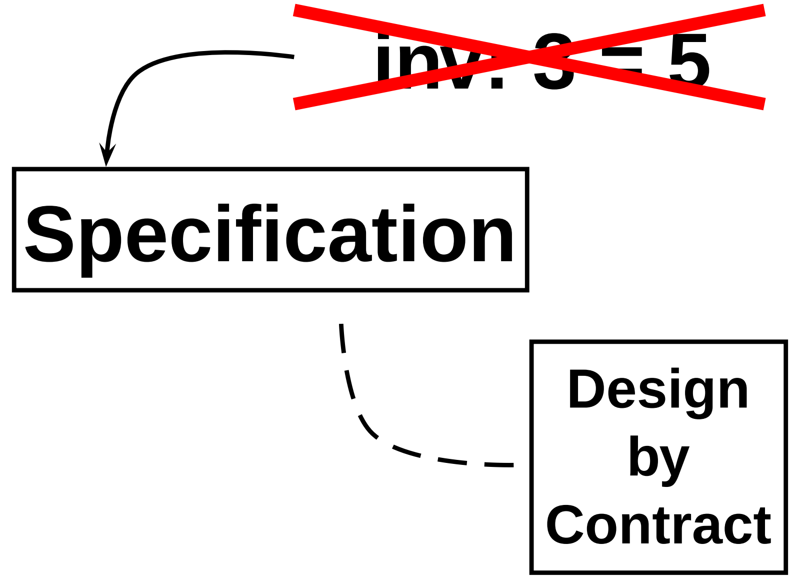
Horizontal Verification

- **Consistency properties**



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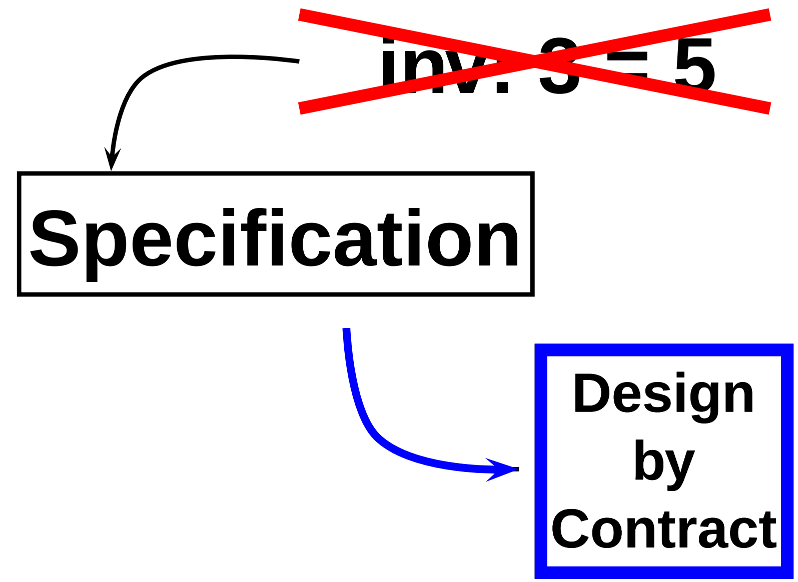
- **Compliance to design principles**



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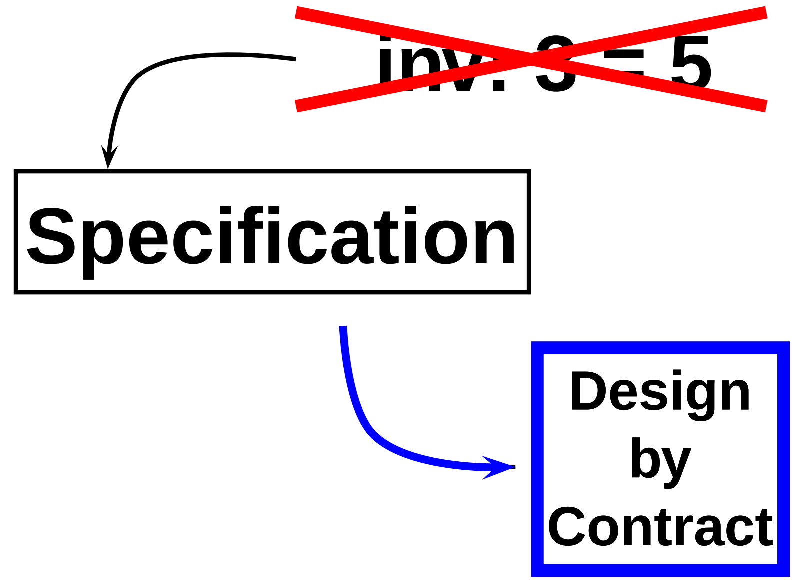
⇒ source code is not involved



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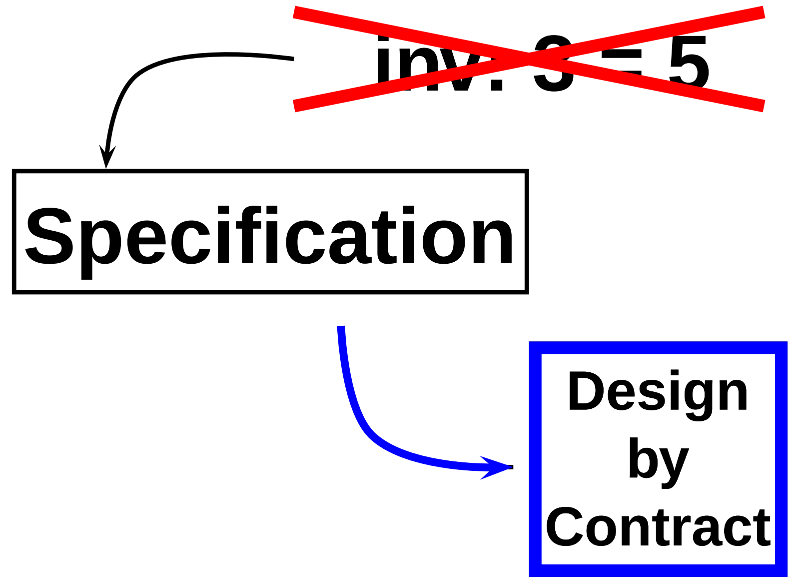

Horizontal Verification

- Consistency properties
- Compliance to design principles

$\Rightarrow$ source code is not involved

Horizontal Verification can be done in Classical First-Order Logic (FOL)

Syntax of Propositional Logic

Signature $\Sigma = (\mathcal{P}, \mathcal{O})$

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Formulas For_0^Σ

- **Propositional Variables are formulas**
- **If** G **and** H **are formulas then**

$$\neg G, (G \wedge H) \text{ and } (G \vee H)$$

are also formulas

Semantics of Propositional Logic

Interpretation (Assignment) I

Assigns a definite truth value to each propositional variable

$$I : \mathcal{P} \rightarrow \{true, false\}$$

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$$val_I : For_0^\Sigma \; \rightarrow \; \{true, false\}$$

$$val_I(P_i) = I(P_i) \qquad val_I(P_i \wedge P_j) = \begin{cases} true & \text{if } val_I(P_i) = true \text{ and} \\ & \quad val_I(P_j) = true \\ false & otherwise \end{cases}$$

$\ldots (and\ so\ on)$

»The truth that's me.«, said the tautology.

Let $\Phi \in For_0^\Sigma$, $\Gamma \subset For_0^\Sigma$

- I **is a model for** Φ **iff.** $val_I(\Phi) = true$ **(write:** $I \models \Phi$**)**

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- **If** Φ **is valid under all interpretations, i.e**

$$\emptyset \models \Phi \ (\textbf{short}: \ \models \Phi)$$

then Φ **is called a** **tautology****.**

Orientation Map

THE SUN SHINES

THE PEOPLE ARE HAPPY

Orientation Map

	THE SUN SHINES	THE PEOPLE ARE HAPPY
Syntax	A	B

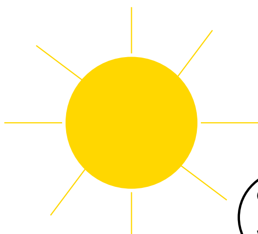
Orientation Map

	IF THE SUN SHINES	THEN	THE PEOPLE ARE HAPPY
Syntax	A	$\longrightarrow$	B

Orientation Map

IF THE SUN SHINES THEN THE PEOPLE ARE HAPPY

Syntax $A \longrightarrow B$

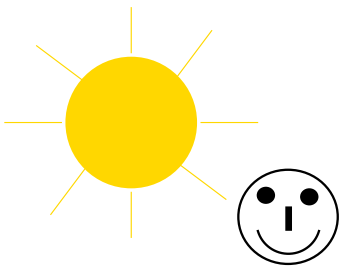
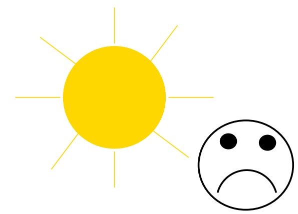


True ☺ ◀ **Semantics** ▶ ☹ **False**

Orientation Map

IF THE SUN SHINES THEN THE PEOPLE ARE HAPPY

Syntax A $\longrightarrow$ B

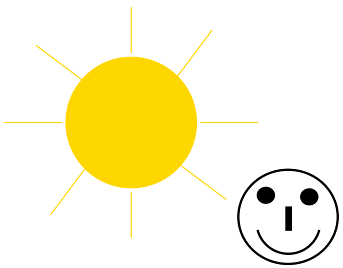
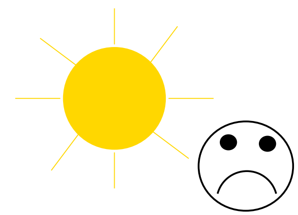
True  ◀ **Semantics** ▶  **False**

Now: Syntactical reasoning

Orientation Map

IF THE SUN SHINES THEN THE PEOPLE ARE HAPPY

Syntax A $\longrightarrow$ B

True  **◂ Semantics ▸**  **False**

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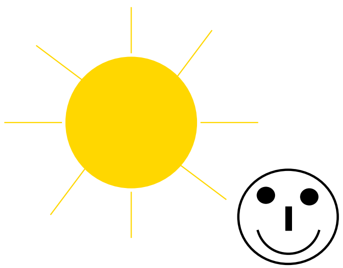
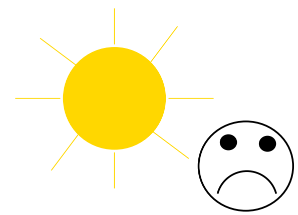
True  ◀ **Semantics** ▶  **False**

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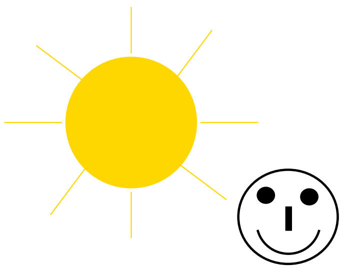
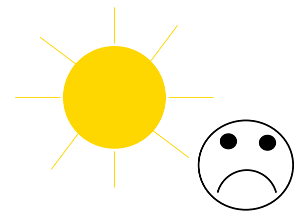
A THE SUN SHINES

$A \rightarrow B$ IF THE SUN SHINES THEN THE PEOPLE ARE HAPPY.

Orientation Map

IF THE SUN SHINES THEN THE PEOPLE ARE HAPPY

Syntax	A	$\longrightarrow$	B

True  ◀ **Semantics** ▶  **False**

Now: Syntactical reasoning

A	THE SUN SHINES
$\underline{A \rightarrow B}$	<u>IF THE SUN SHINES THEN THE PEOPLE ARE HAPPY.</u>
B	THE PEOPLE ARE HAPPY

A Bridge between Semantics and Syntax

Deduction Theorem

Let $\Gamma \subset For_\Sigma$**,** $\Phi, \Psi \in For_\Sigma$

$$\Gamma, \Psi \models \Phi \textbf{ iff. } \Gamma \models \Psi \to \Phi$$

Establishes a relationship between the semantical consequence '$\models$'

and the syntactical implication '$\to$'

Reasoning as Syntactical Transformations

Task: Compute $\Gamma \models \Phi$ by performing syntactical transformations

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Sequent Calculus '$\Longrightarrow$':

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Sequent Calculus '$\Longrightarrow$':

$$\underbrace{\psi_1, \ldots, \psi_n}_{Premises} \Longrightarrow \underbrace{\phi_1, \ldots, \phi_n}_{Consequences}$$

has the same semantic as

$$\psi_1 \wedge \ldots \wedge \psi_n \rightarrow \phi_1 \vee \ldots \vee \phi_n$$

Rules of the Sequent Calculus

	left side	right side
not	$\dfrac{\Gamma \implies A, \Delta}{\Gamma, \neg A \implies \Delta}$	$\dfrac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta}$

Rules of the Sequent Calculus

	left side	right side
not	$\dfrac{\Gamma \implies A, \Delta}{\Gamma, \neg A \implies \Delta}$	$\dfrac{\Gamma, A \implies \Delta}{\Gamma \implies \neg A, \Delta}$
and	$\dfrac{\Gamma, A, B \implies \Delta}{\Gamma, A \wedge B \implies \Delta}$	$\dfrac{\Gamma \implies A, \Delta \qquad \Gamma \implies B, \Delta}{\Gamma \implies A \wedge B, \Delta}$

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CLOSE(AXIOM) $\dfrac{*}{\Gamma, A \Longrightarrow A, \Delta}$

Proof of Modus Ponens

$$\Gamma \implies (A \land (A \to B)) \to B, \Delta$$

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$$\Gamma, (A \wedge (A \to B)) \implies B, \Delta$$

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$$\frac{\dfrac{\dfrac{\overline{} \qquad \overline{}}{\Gamma, A, (A \to B) \Longrightarrow B, \Delta}}{\Gamma, (A \land (A \to B)) \Longrightarrow B, \Delta}}{\Gamma \Longrightarrow (A \land (A \to B)) \to B, \Delta}$$

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A proof is closed, if all its goals are closed.

Propositional logic is insufficient

A ALL PERSONS ARE HAPPY

Propositional logic is insufficient

A	**ALL PERSONS ARE HAPPY**
B	**PAT IS A PERSON**

Propositional logic is insufficient

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$?$	PAT IS HAPPY

Propositional Logic lacks a possibility to talk about individuals.

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Propositional Logic lacks a possibility to talk about individuals.

$\Rightarrow$ First-Order Logic (FOL)

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Signature $\Sigma = (\mathcal{P}, \mathcal{F}, \mathcal{V}, \alpha, \mathcal{O} \cup \mathcal{Q} \cup \{\doteq\})$

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Additional: Let t_1, t_2 **be terms then** $t_1 \doteq t_2$ **is a formula.**

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Interpretation $\mathcal{D}$=(U, I):

U **is the non-empty universe**

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Interpretation $\mathcal{D}$=(U, I):

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  $P^I \subseteq \{(x_1, \ldots, x_n) | x_i \in U, n = \alpha(P)\}$

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$$val_{\mathcal{D},\beta}(\forall x.\Phi(x)) = \begin{cases} true & \textbf{for all } d \in U : val_{\mathcal{D},\beta_x^d}(\Phi) = true \\ false & otherwise \end{cases}$$

Definitions

Satisfiability, Model and Universal validity

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REMARK: Sorted First-Order Logic

Variables and functions is given a sort $\in Sorts$

$\forall x : S.\Phi(x)$ **i.e.** $\forall x.(S(x) \rightarrow \Phi(x))$

$\exists x : S.\Phi(x)$ **i.e.** $\exists x.(S(x) \wedge \Phi(x))$

Do we have a deduction theorem at hand?

$$\Gamma, \Psi \models \Phi \text{ **iff.** } \Gamma \models \Psi \to \Phi$$

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From now on only closed formulas are considered.

Sequent Calculus for FOL

	left side	right side

- $t \in Term_\Sigma$ **an arbitrary ground term (no variables)**
- $c\ new$ **constant**

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insert eq.	$\dfrac{\Gamma, x \doteq y \implies \{x/y\}\Phi(x), \Delta}{\Gamma, x \doteq y \implies \Phi(x), \Delta}$	—

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Explaining the Rules (I)

The following description shall explain the first-order calculus rules on an intuitive (informal) level. For the remainding section all mentioned terms are ground terms, this means they contain no variables.

`all left` If a $\forall x.\Phi(x)$ occurs in the premise, one can add an instantiation with an arbitrary term t to the premises. This is sound as $\{x/t\}\Phi(x)$ holds for all elements of the universe, in particular for the element t is evaluated to. In contrast to the former rules one keeps the quantified formula in the antecedent as one may require more than one instantiation.

`ex. left` $\exists x.\Phi(x)$ can be replaced by $\{x/c\}\Phi$ where c is a new constant. c is thought to be evaluated to the element for which $\Phi(x)$ holds. An already existing term t must not be used as its value is already fixed but in general not to the element satisfying $\Phi(x)$.

Explaining the Rules (II)

`all right` A common way to show that $\forall x.\Phi(x)$ holds, is to take an element of an arbitrary value. In other words, if $\{x/c\}\Phi(x)$ can be shown for a new constant c then the result can be generalised, as no assumptions about the value of c have been made.
In contrast, the generalisation is not possible if an already existing term t is used instead. The value of t has been already fixed to a certain value, which may randomly satisfy $\Phi(x)$, but this may not necessarily be the case for all other elements of the universe (similar to: $2, 3, 5, 7$ are primes, so all odd numbers are primes).

`ex. right` If $\exists x.\Phi(x)$ has to be proven, one can try to prove it for an arbitrary term t. If one uses the wrong term t, this means a term for which $\Phi(x)$ is *false* it is not worse, one only gets *false* on the right side, which is the neutral element of $\vee$ and so it can just be removed from the sequent. The existential quantified formula is not removed from the sequent, so that one can try to prove the formula for another term t' (sometimes one even has to instantiate the existential quantifiers and all instances are required).

Example

DEMO

Towards Program Verification

Vertical Verification

- **Prove that the implementation fulfills the specification (equivalence for complete specifications)**

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In contrast to testing,
verification can show the absence of errors

Do we really need another kind of logics?

»There is a tradition in logic, carried over into computer science, to think of pure first order logic as a universal language.
In fact first order language is about as useful in verification as a Turing machine is in software engineering:

CUTE TO WATCH BUT NOT VERY USEFUL.«

V. Pratt

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$\Rightarrow$ **Reasoning about programs must consider the current program state.**

Dynamic Logics for a simple 'while' language

Signature

$\Sigma = (\mathcal{P}, \mathcal{F}, \Pi_0, \mathcal{O} \cup \{\langle\cdot\rangle, [.]\})$, $Sorts = \{\texttt{int}, \texttt{boolean}\}$

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are programs in Π**.**

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Definition of Terms

Defined as in first-order logics. But we distinct between

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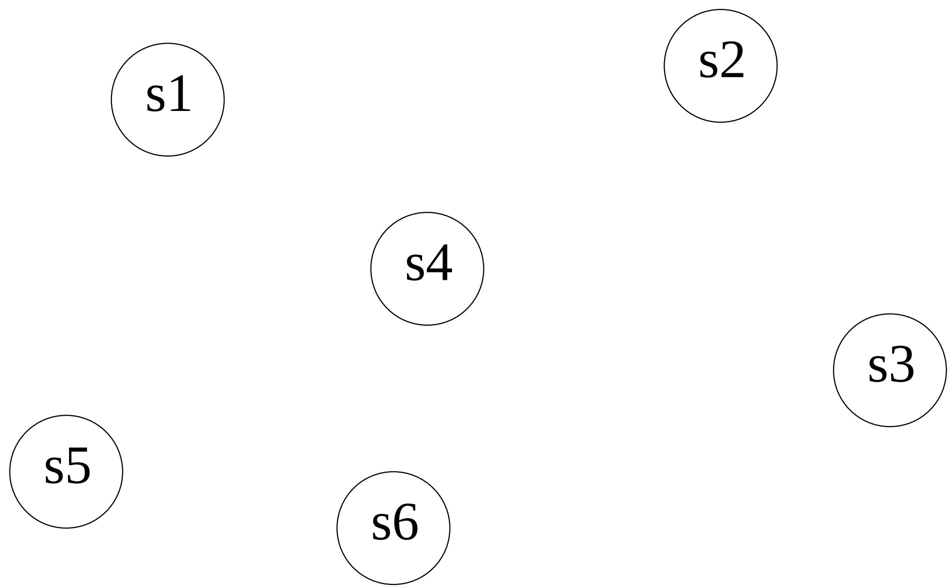
Semantics of Dynamic Logic - Kripke Structure

Kripke-Structure $\mathcal{K} = (States, \rho)$

where $s \in State, s = (\mathcal{U}, I)$ **and** $\rho : \Pi_0 \rightarrow States \times States$



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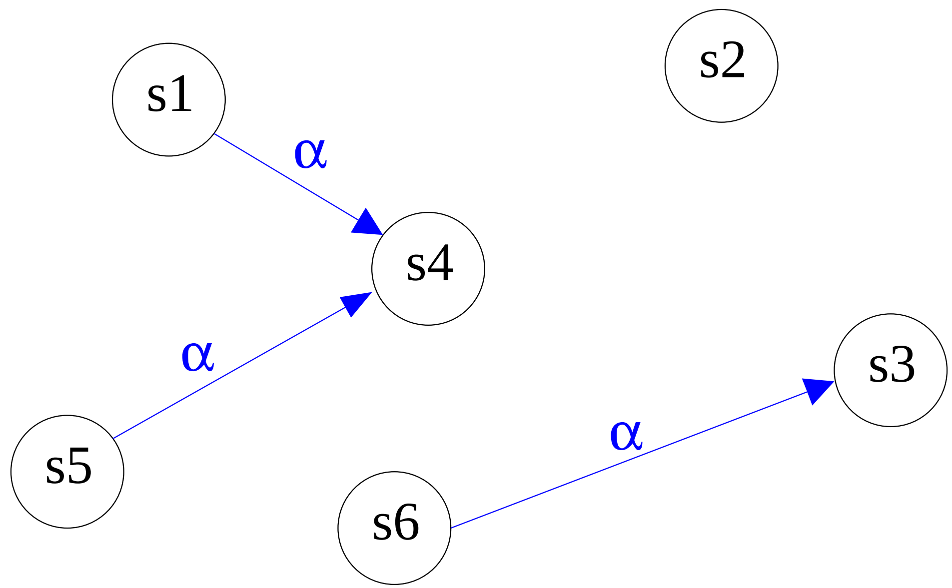

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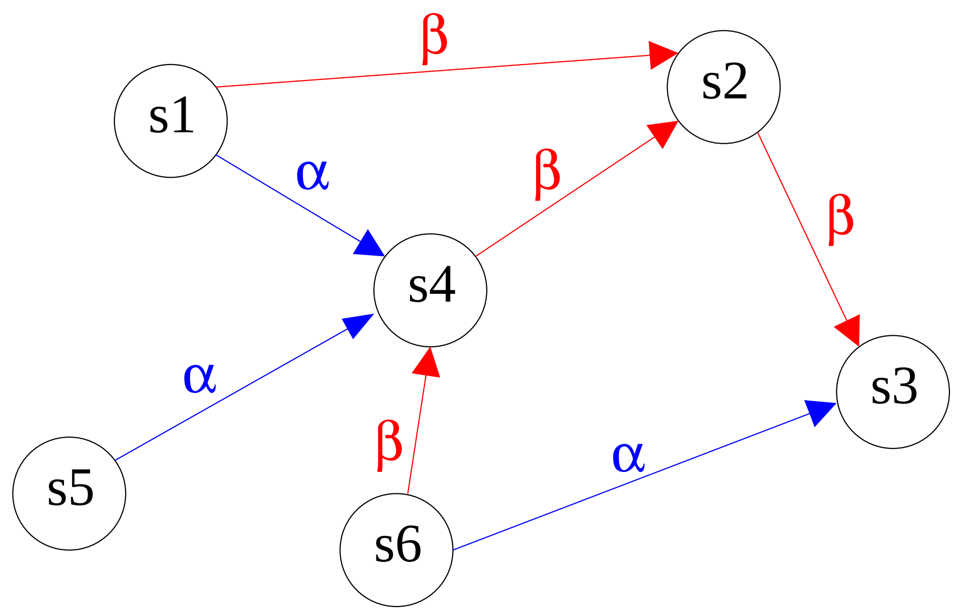

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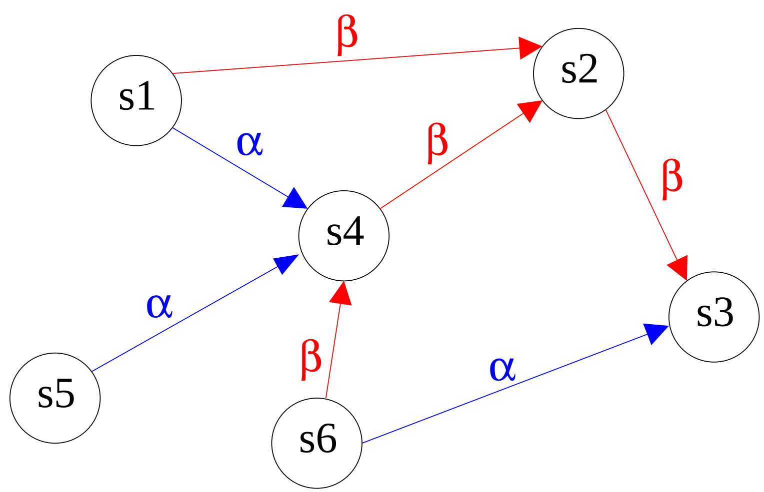
$\langle\alpha\rangle\Phi$ **There exists an α-reachable state, such that Φ holds.**



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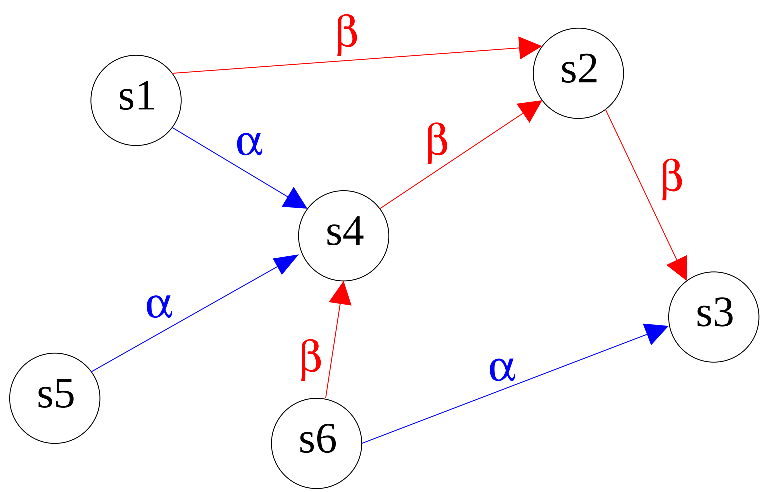

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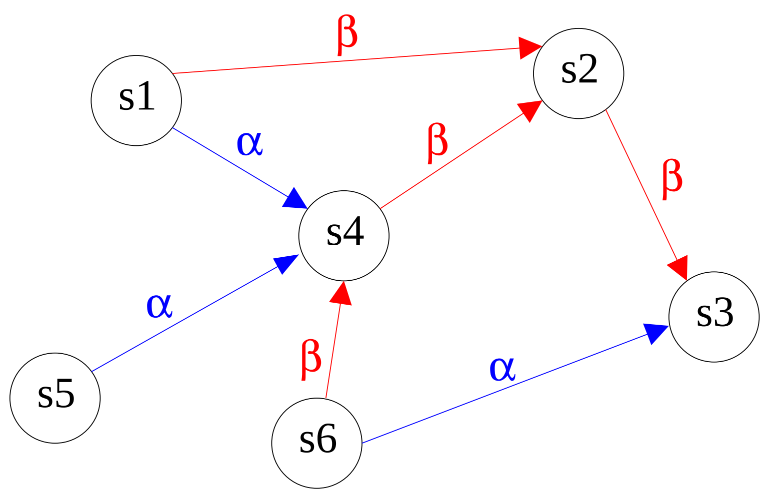

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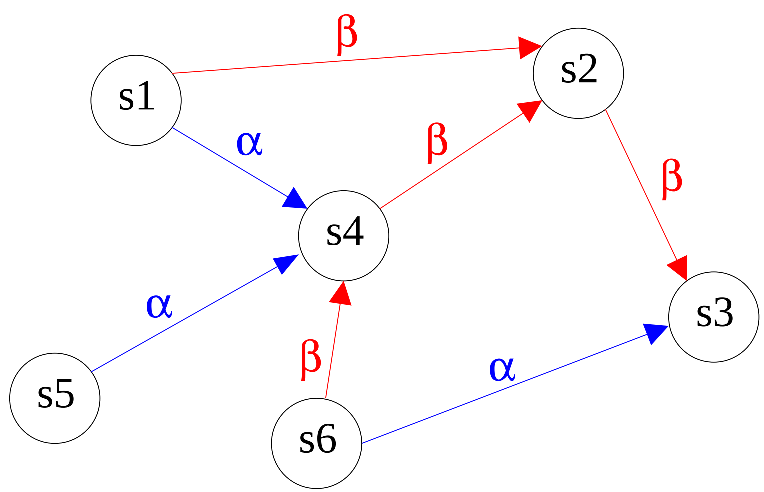

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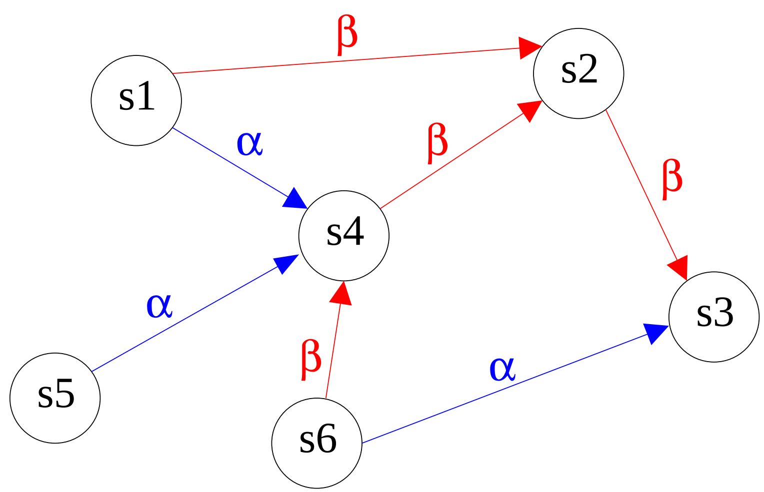

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<u>Duality:</u> $\langle\alpha\rangle\Phi$ iff. $\neg[\alpha]\neg\Phi$

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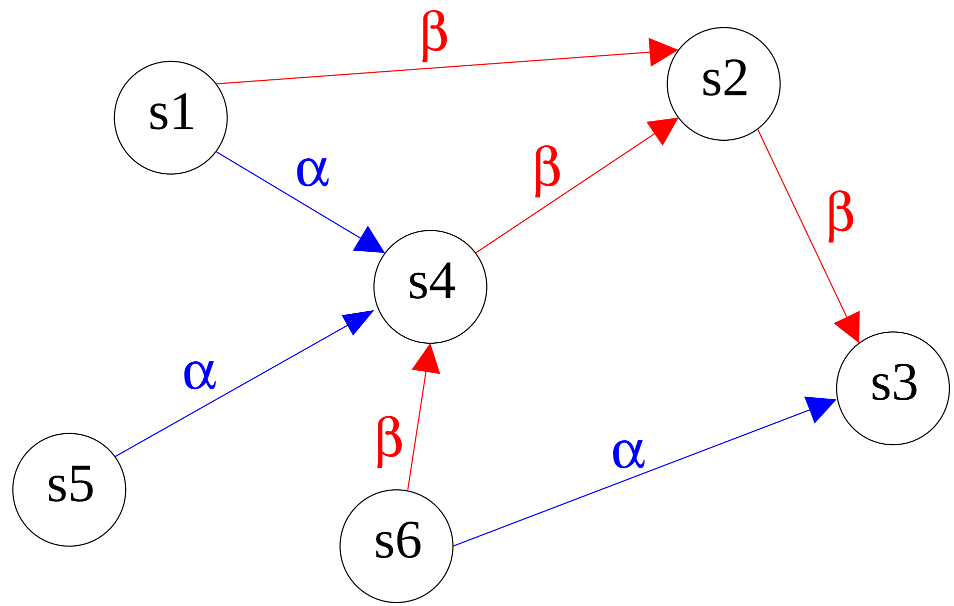
Let $\mathcal{P} = \{A, B, C\}$, $\mathcal{D} = I\!N$ **and**

$s1 : I = \{A, B\}$, $s2 : I = \{C\}$, $s4 : I = \{A\}$



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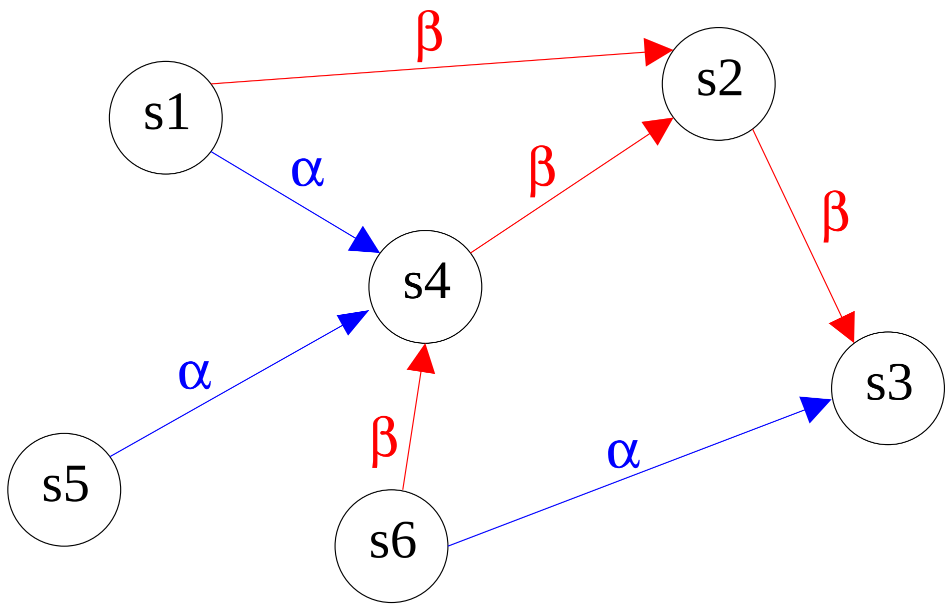

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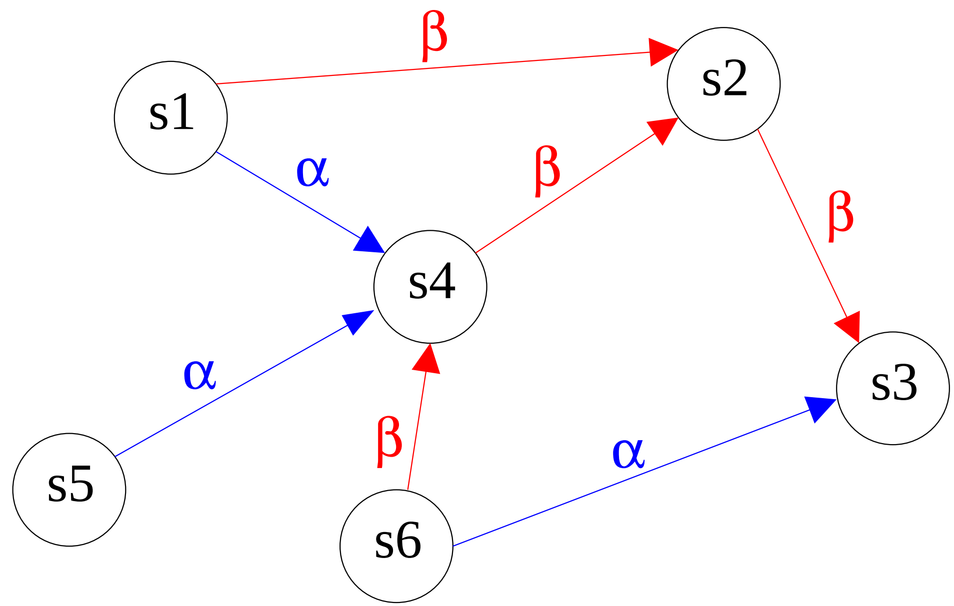

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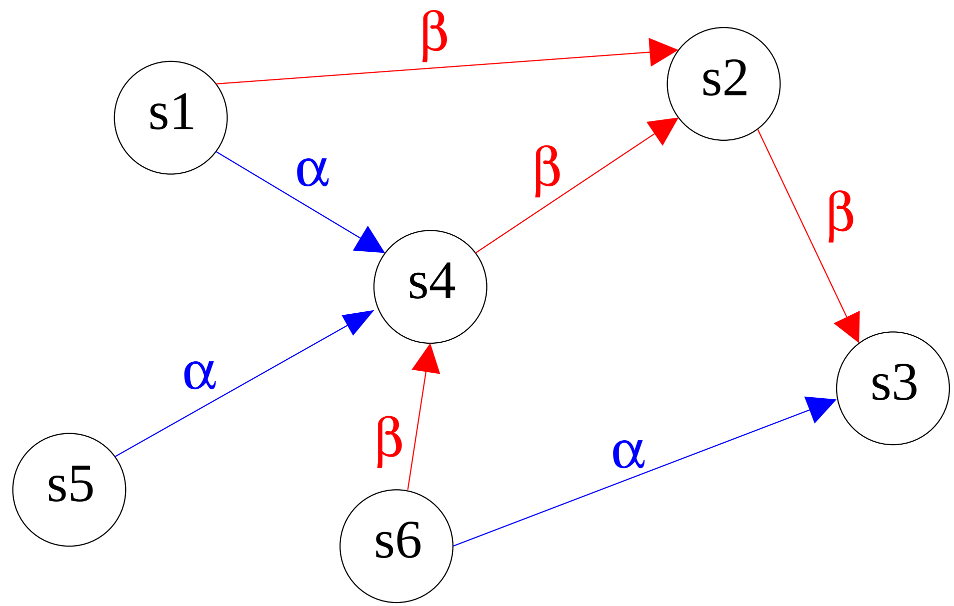

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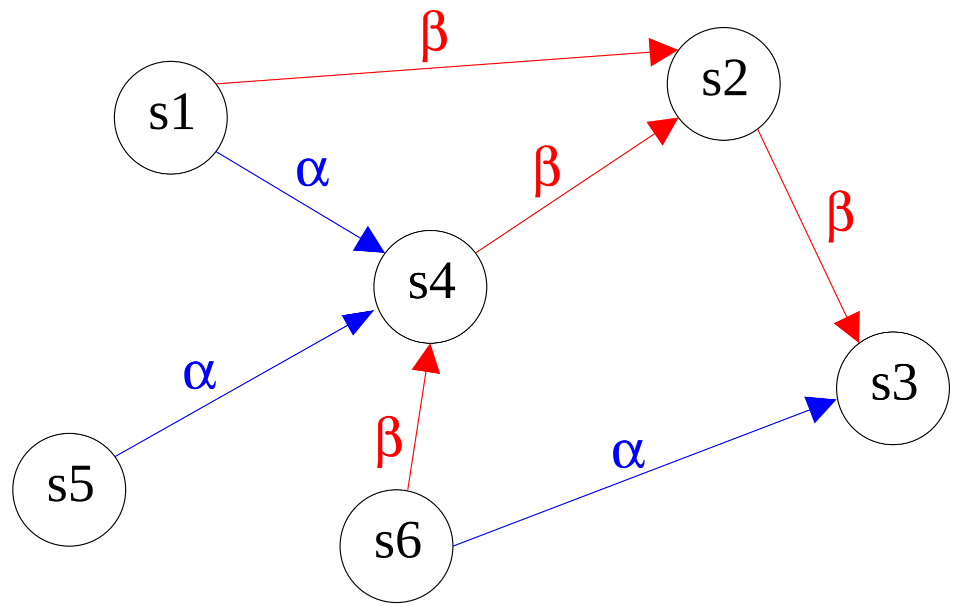

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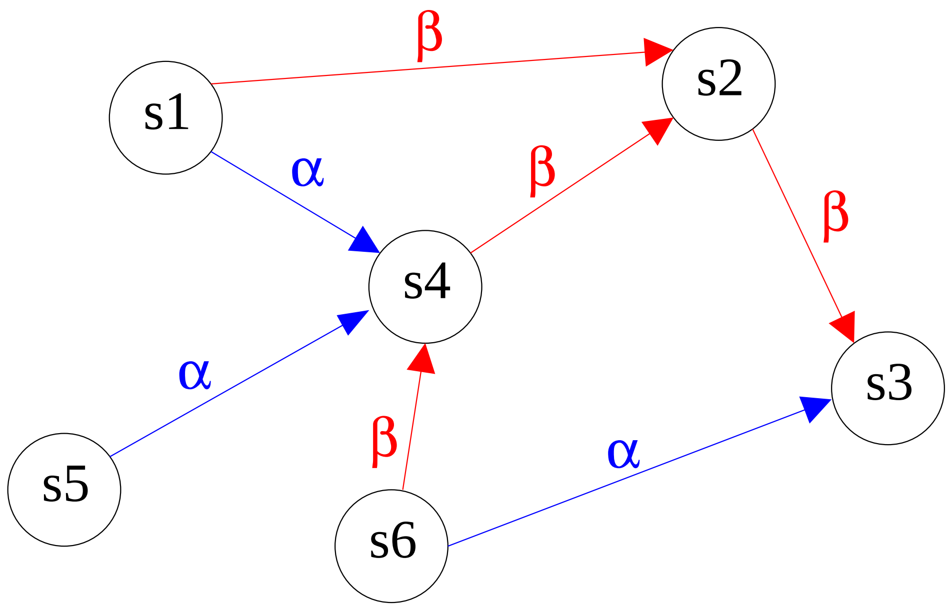

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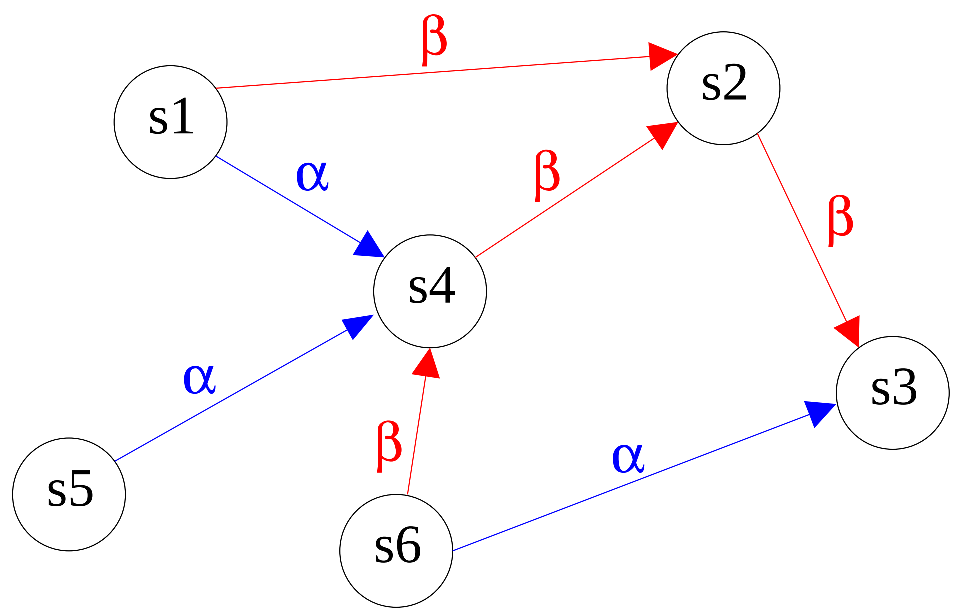

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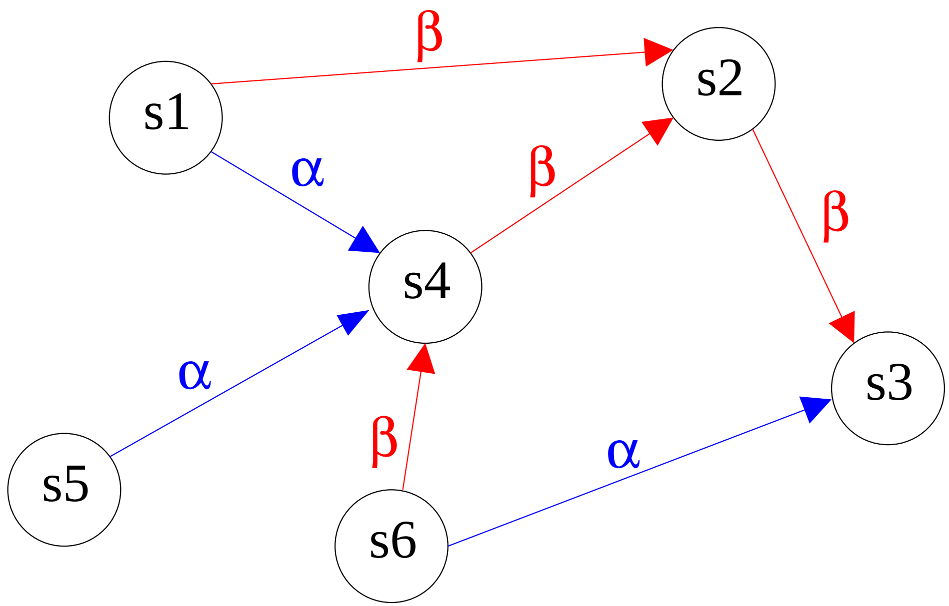

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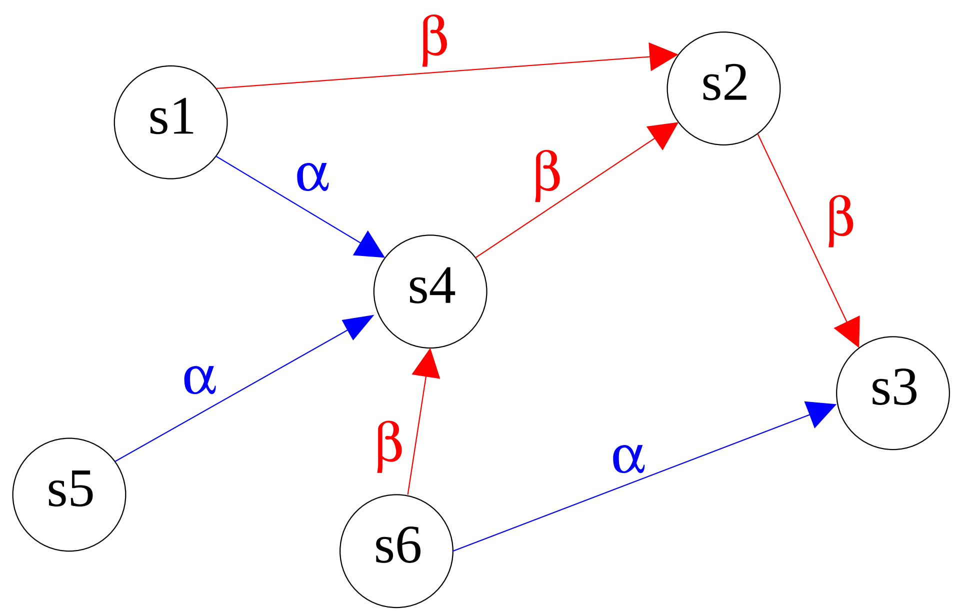

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Example

```
y=1;
x=3;
while (x>0) {
  y=y*x;
  x=x-1;
}
```

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States $s = (U, I, \sigma)$

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Further agreement:

- **Logic variables vs. program variables:**

  Program variables cannot be quantified. Their value depends on the current state. Therefore each state contains a function

  $\sigma : ProgVar \to U$**.**

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Further agreement:

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  Program variables cannot be quantified. Their value depends on the current state. Therefore each state contains a function

  $\sigma : ProgVar \to U$.

  On the other hand, logic variables are not allowed to occur in programs and they must be bound by a quantifier.

Local Validity

There is some choice selecting the consequence relation $\models$.

The deduction theorem holds for the <u>local</u> version:

$$\Gamma \models \Phi$$

iff.

for all states g: if $g \models \Gamma$ then $g \models \Phi$

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(global version:

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iff.

for all states g: $g \models \Gamma$ then for all states g: $g \models \Phi$

)

Sequent Calculus Rules

$$\textbf{IF-ELSE}\ \frac{\Gamma, b \doteq true \implies \langle\alpha\rangle\Phi, \Delta \qquad \Gamma \implies b \doteq true, \langle\beta\rangle\Phi, \Delta}{\Gamma \implies \langle \texttt{if}\ (b)\ \texttt{then}\ \alpha;\ \texttt{else}\ \beta;\rangle\Phi, \Delta}$$

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Assignment $$\frac{\Gamma^{x \leftarrow y}, x \doteq t \quad \vdash \quad \Phi,\ \Delta^{x \leftarrow y}}{\Gamma \quad \vdash \quad \langle x = t\rangle\Phi,\ \Delta}\ (y\ new\ variable)$$

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$$\frac{\Gamma^{x\leftarrow y}, x \doteq t \vdash \Phi,\ \Delta^{x\leftarrow y}}{\Gamma \vdash \langle x = t\rangle\Phi,\ \Delta}\ (y\ new\ variable)$$

Example

DEMO