Introduction to Artificial Intelligence

First-order Logic

(Logic, Deduction, Knowledge Representation)

Bernhard Beckert

UNIVERSITÄT KOBLENZ-LANDAU

Winter Term 2004/2005

Outline

- Why first-order logic?
- Syntax and semantics of first-order logic
- Fun with sentences
- Wumpus world in first-order logic

Pros and Cons of Propositional Logic

- 😆 **Propositional logic is declarative:**
  pieces of syntax correspond to facts

Pros and Cons of Propositional Logic

- Propositional logic is declarative:
  pieces of syntax correspond to facts

- Propositional logic allows partial / disjunctive / negated information
  (unlike most data structures and databases)

Pros and Cons of Propositional Logic

- **Propositional logic is declarative:**
  pieces of syntax correspond to facts

- **Propositional logic allows partial / disjunctive / negated information**
  (unlike most data structures and databases)

- **Propositional logic is compositional:**
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

Pros and Cons of Propositional Logic

- Propositional logic is **declarative**:
  pieces of syntax correspond to facts

- Propositional logic allows partial / disjunctive / negated information
  (unlike most data structures and databases)

- Propositional logic is **compositional**:
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

- Meaning in propositional logic is **context-independent**
  (unlike natural language, where meaning depends on context)

Pros and Cons of Propositional Logic

- **Propositional logic is declarative:**
  pieces of syntax correspond to facts

- **Propositional logic allows partial / disjunctive / negated information**
  (unlike most data structures and databases)

- **Propositional logic is compositional:**
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

- **Meaning in propositional logic is context-independent**
  (unlike natural language, where meaning depends on context)

- **Propositional logic has very limited expressive power**
  (unlike natural language)

  Example:
  Cannot say “pits cause breezes in adjacent squares”
  except by writing one sentence for each square

First-order Logic

Propositional logic

Assumes that the world contains facts

First-order Logic

Propositional logic

Assumes that the world contains facts

First-order logic

Assumes that the world contains

- **Objects**
  people, houses, numbers, theories, Donald Duck, colors, centuries, ...

First-order Logic

Propositional logic

Assumes that the world contains facts

First-order logic

Assumes that the world contains

- **Objects**
  people, houses, numbers, theories, Donald Duck, colors, centuries, ...
- **Relations**
  red, round, prime, multistoried, ...
  brother of, bigger than, part of, has color, occurred after, owns, ...

First-order Logic

Propositional logic

Assumes that the world contains facts

First-order logic

Assumes that the world contains

- **Objects**
  people, houses, numbers, theories, Donald Duck, colors, centuries, . . .
- **Relations**
  red, round, prime, multistoried, . . .
  brother of, bigger than, part of, has color, occurred after, owns, . . .
- **Functions**
  $+$**, middle of, father of, one more than, beginning of, . . .**

Syntax of First-order Logic: Basic Elements

Symbols

Constants	$\mathit{KingJohn}$, 2, $\mathit{Koblenz}$, $C, \ldots$
Predicates	$\mathit{Brother}$, $>$, $=$, $\ldots$
Functions	Sqrt, $\mathit{LeftLegOf}$, $\ldots$
Variables	x, y, a, b, $\ldots$
Connectives	$\land \quad \lor \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
Quantifiers	$\forall \quad \exists$

Syntax of First-order Logic: Basic Elements

Symbols

Constants	$KingJohn,\ 2,\ Koblenz,\ C, \ldots$
Predicates	$Brother,\ >,\ =,\ \ldots$
Functions	$Sqrt,\ LeftLegOf,\ \ldots$
Variables	$x,\ y,\ a,\ b,\ \ldots$
Connectives	$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
Quantifiers	$\forall \quad \exists$

Note

The equality predicate is always in the vocabulary

It is written in infix notation

Syntax of First-order Logic: Atomic Sentences

Atomic sentence

$\textit{predicate}\,(\,\textit{term}_1,\ \ldots,\ \textit{term}_n\,)$

or

$\textit{term}_1 = \textit{term}_2$

Syntax of First-order Logic: Atomic Sentences

Atomic sentence

$\textit{predicate}\,(\ \textit{term}_1,\ \ldots,\ \textit{term}_n\ )$

or

$\textit{term}_1 = \textit{term}_2$

Term

$\textit{function}\,(\ \textit{term}_1,\ \ldots,\ \textit{term}_n\ )$

or

constant

or

variable

Syntax of First-order Logic: Atomic Sentences

Example

$$\mathit{Brother}\,(\,\mathit{KingJohn},\;\mathit{RichardTheLionheart}\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{\mathit{Brother}}_{\text{predicate}}\,(\,\underbrace{\mathit{KingJohn}}_{\text{constant}},\ \underbrace{\mathit{RichardTheLionheart}}_{\text{constant}}\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{Brother}_{\textbf{predicate}}\ (\ \underbrace{\underbrace{KingJohn}_{\textbf{constant}}}_{\textbf{term}},\ \underbrace{\underbrace{RichardTheLionheart}_{\textbf{constant}}}_{\textbf{term}}\ )$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{\underbrace{Brother}_{\text{predicate}}\,(\,\underbrace{\underbrace{KingJohn}_{\text{constant}}}_{\text{term}},\ \underbrace{\underbrace{RichardTheLionheart}_{\text{constant}}}_{\text{term}}\,)}_{\text{atomic sentence}}$$

Syntax of First-order Logic: Atomic Sentences

Example

$$>\quad (\, \mathit{Length}(\mathit{LeftLegOf}(\mathit{Richard})),\ \mathit{Length}(\mathit{LeftLegOf}(\mathit{KingJohn}))\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{>}_{\textbf{predicate}} \; (\,\underbrace{Length}_{\textbf{function}}(\underbrace{LeftLegOf}_{\textbf{function}}(\underbrace{Richard}_{\textbf{constant}})),\; \underbrace{Length}_{\textbf{function}}(\underbrace{LeftLegOf}_{\textbf{function}}(\underbrace{KingJohn}_{\textbf{constant}}))\,)$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{>}_{\textsf{predicate}} \quad (\underbrace{\underbrace{Length}_{\textsf{function}}(\underbrace{LeftLegOf}_{\textsf{function}}(\underbrace{Richard}_{\textsf{constant}}))}_{\textsf{term}}, \underbrace{\underbrace{Length}_{\textsf{function}}(\underbrace{LeftLegOf}_{\textsf{function}}(\underbrace{KingJohn}_{\textsf{constant}}))}_{\textsf{term}})$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{\underbrace{>}_{\text{predicate}} \quad (\underbrace{\underbrace{Length}_{\text{function}}(\underbrace{LeftLegOf}_{\text{function}}(\underbrace{Richard}_{\text{constant}}))}_{\text{term}}, \underbrace{\underbrace{Length}_{\text{function}}(\underbrace{LeftLegOf}_{\text{function}}(\underbrace{KingJohn}_{\text{constant}}))}_{\text{term}} \;)}_{\text{atomic sentence}}$$

Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \wedge S_2 \qquad S_1 \vee S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \wedge S_2 \qquad S_1 \vee S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

Example

$$Sibling(\,KingJohn, Richard\,) \quad \Rightarrow \quad Sibling(\,Richard, KingJohn\,)$$

Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \wedge S_2 \qquad S_1 \vee S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

Example

$$\underbrace{Sibling}_{\text{predicate}}(\underbrace{KingJohn}_{\text{term}}, \underbrace{Richard}_{\text{term}}) \Rightarrow \underbrace{Sibling}_{\text{predicate}}(\underbrace{Richard}_{\text{term}}, \underbrace{KingJohn}_{\text{term}})$$

Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \wedge S_2 \qquad S_1 \vee S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

Example

$$\underbrace{\underbrace{Sibling}_{\text{predicate}}(\underbrace{KingJohn}_{\text{term}},\underbrace{Richard}_{\text{term}})}_{\text{atomic sentence}} \Rightarrow \underbrace{\underbrace{Sibling}_{\text{predicate}}(\underbrace{Richard}_{\text{term}},\underbrace{KingJohn}_{\text{term}})}_{\text{atomic sentence}}$$

Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \land S_2 \qquad S_1 \lor S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

Example

$$\underbrace{\underbrace{\underbrace{Sibling}_{\text{predicate}}(\underbrace{KingJohn}_{\text{term}},\underbrace{Richard}_{\text{term}})}_{\text{atomic sentence}} \Rightarrow \underbrace{\underbrace{Sibling}_{\text{predicate}}(\underbrace{Richard}_{\text{term}},\underbrace{KingJohn}_{\text{term}})}_{\text{atomic sentence}}}_{\text{complex sentence}}$$

Semantics in First-order Logic

Models of first-order logic

Sentences are true or false with respect to models, which consist of

- **a domain (also called universe)**
- **an interpretation**

Semantics in First-order Logic

Models of first-order logic

Sentences are true or false with respect to models, which consist of

- **a domain (also called universe)**
- **an interpretation**

Domain

A non-empty (finite or infinite) set of arbitrary elements

Semantics in First-order Logic

Models of first-order logic

Sentences are true or false with respect to models, which consist of

– a domain (also called universe)
– an interpretation

Domain

A non-empty (finite or infinite) set of arbitrary elements

Interpretation

Assigns to each

– constant symbol: a domain element

– predicate symbol: a relation on the domain (of appropriate arity)

– function symbol: a function on the domain (of appropriate arity)

Semantics in First-order Logic

Definition

An atomic sentence

$$\textit{predicate} \, (\, \textit{term}_1, \, \ldots, \, \textit{term}_n \,)$$

is true in a certain model (that consists of a domain and an interpretation)

iff

the domain elements that are the interpretations of $term_1, \ldots, term_n$
are in the relation that is the interpretation of *predicate*

Semantics in First-order Logic

Definition

An atomic sentence

$$\textit{predicate}\ (\ \textit{term}_1,\ \ldots,\ \textit{term}_n\ )$$

is true in a certain model (that consists of a domain and an interpretation)

iff

the domain elements that are the interpretations of $term_1, \ldots, term_n$
are in the relation that is the interpretation of *predicate*

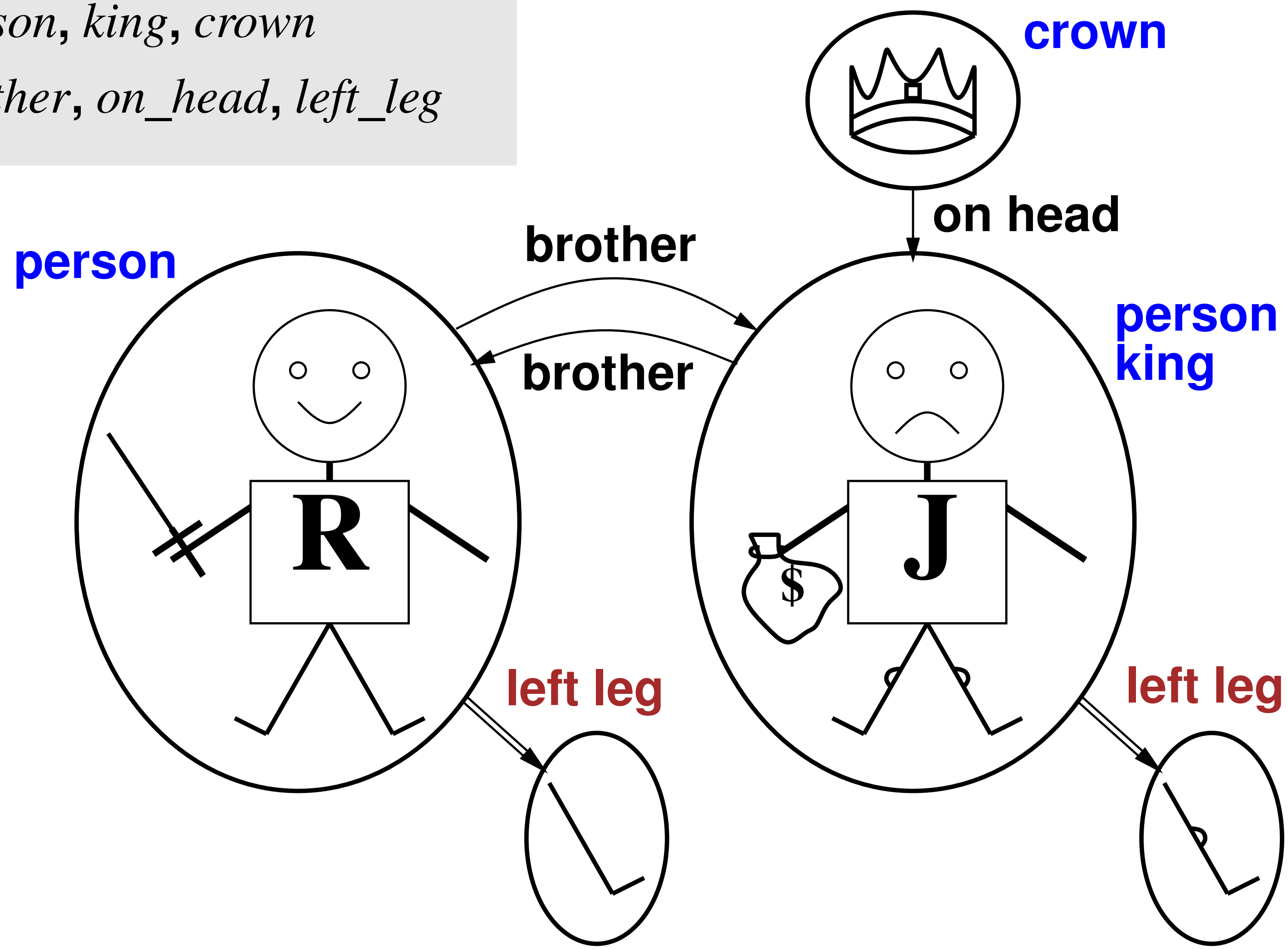
The truth value of a complex sentence in a model
is computed from the truth-values of its atomic sub-sentences
in the same way as in propositional logic

Models for First-order Logic: Example

Constants: *KingJohn*, *Richard*

Predicates: *person*, *king*, *crown*

Functions: *brother*, *on_head*, *left_leg*



Universal Quantification: Syntax

Syntax

$\forall$ *variables sentence*

Universal Quantification: Syntax

Syntax

$\forall$ *variables sentence*

Example

“Everyone studying in Koblenz is smart:

$$\forall \underbrace{\quad x \quad}_{\text{variables}} \underbrace{(StudiesAt(x, Koblenz) \Rightarrow Smart(x))}_{\text{sentence}}$$

Universal Quantification: Semantics

Semantics

$\forall x P$ **is true in a model**

iff

for all domain elements d **in the model:**
P **is true in the model when** x **is interpreted by** d

Universal Quantification: Semantics

Semantics

$\forall x P$ is true in a model

iff

for all domain elements d in the model:
P is true in the model when x is interpreted by d

Intuition

$\forall x P$ is roughly equivalent to the conjunction of all instances of P

Universal Quantification: Semantics

Semantics

$\forall x P$ **is true in a model**

iff

for all domain elements d **in the model:**
P **is true in the model when** x **is interpreted by** d

Intuition

$\forall x P$ **is roughly equivalent to the conjunction of all instances of** P

Example $\forall x\, StudiesAt(x, Koblenz) \Rightarrow Smart(x)$ **equivalent to:**

$$
\begin{array}{ll}
 & StudiesAt(KingJohn, Koblenz) \Rightarrow Smart(KingJohn) \\
\wedge & StudiesAt(Richard, Koblenz) \Rightarrow Smart(Richard) \\
\wedge & StudiesAt(Koblenz, Koblenz) \Rightarrow Smart(Koblenz) \\
\wedge & \ldots
\end{array}
$$

A Common Mistake to Avoid

Note

$\Rightarrow$ **is the main connective with** $\forall$

Common mistake

Using $\land$ **as the main connective with** $\forall$

A Common Mistake to Avoid

Note

$\Rightarrow$ **is the main connective with** $\forall$

Common mistake

Using $\land$ **as the main connective with** $\forall$

Example

Correct: $\forall x\,(StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

"Everyone who studies at Koblenz is smart"

A Common Mistake to Avoid

Note

$\Rightarrow$ **is the main connective with** $\forall$

Common mistake

Using $\wedge$ **as the main connective with** $\forall$

Example

Correct: $\forall x\,(StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

“Everyone who studies at Koblenz is smart”

Wrong: $\forall x\,(StudiesAt(x, Koblenz) \wedge Smart(x))$

“Everyone studies at Koblenz and is smart”, i.e.,

“Everyone studies at Koblenz and everyone is smart”

Existential Quantification: Syntax

Syntax

$\exists$ *variables sentence*

Existential Quantification: Syntax

Syntax

$\exists$ *variables sentence*

Example

"Someone studying in Landau is smart:

$$\exists \underbrace{x}_{\text{variables}} \underbrace{(StudiesAt(x, Landau) \land Smart(x))}_{\text{sentence}}$$

Existential Quantification: Semantics

Semantics

$\exists x P$ is true in a model

iff

there is a domain element d in the model such that:
P is true in the model when x is interpreted by d

Existential Quantification: Semantics

Semantics

$\exists x P$ **is true in a model**

iff

there is a domain element d in the model such that:
P is true in the model when x is interpreted by d

Intuition

$\exists x P$ **is roughly equivalent to the disjunction of all instances of P**

Existential Quantification: Semantics

Semantics

$\exists x P$ **is true in a model**

iff

there is a domain element d in the model such that:
P is true in the model when x is interpreted by d

Intuition

$\exists x P$ **is roughly equivalent to the disjunction of all instances of P**

Example $\exists\, x\, StudiesAt(x, Landau) \wedge Smart(x)$ **equivalent to:**

$$
\begin{array}{ll}
 & StudiesAt(KingJohn, Landau) \wedge Smart(KingJohn) \\
\vee & StudiesAt(Richard, Landau) \wedge Smart(Richard) \\
\vee & StudiesAt(Landau, Landau) \wedge Smart(Landau) \\
\vee & \ldots
\end{array}
$$

Another Common Mistake to Avoid

Note

$\wedge$ **is the main connective with** $\exists$

Common mistake

Using $\Rightarrow$ **as the main connective with** $\exists$

Another Common Mistake to Avoid

Note

$\land$ **is the main connective with** $\exists$

Common mistake

Using $\Rightarrow$ **as the main connective with** $\exists$

Example

Correct: $\exists x\,(StudiesAt(x, Landau) \land Smart(x))$

"There is someone who studies at Landau and is smart"

Another Common Mistake to Avoid

Note

$\wedge$ **is the main connective with** $\exists$

Common mistake

Using $\Rightarrow$ **as the main connective with** $\exists$

Example

Correct: $\exists x\,(StudiesAt(x, Landau) \wedge Smart(x))$

"There is someone who studies at Landau and is smart"

Wrong: $\exists x\,(StudiesAt(x, Landau) \Rightarrow Smart(x))$

"There is someone who, if he/she studies at Landau, is smart"

This is true if there is anyone not studying at Landau

Properties of Quantifiers

Quantifiers of same type commute

$\forall x \forall y$	**is the same as**	$\forall y \forall x$
$\exists x \exists y$	**is the same as**	$\exists y \exists x$

Properties of Quantifiers

Quantifiers of different type do NOT commute

$\exists x \forall y$ **is not the same as** $\forall y \exists x$

Example

$\exists x \forall y Loves(x, y)$

"There is a person who loves everyone in the world"

$\forall y \exists x Loves(x, y)$

"Everyone in the world is loved by at least one person"

(Both hopefully true but different)

Properties of Quantifiers

Quantifiers of different type do NOT commute

$\exists x \forall y$ **is not the same as** $\forall y \exists x$

Example

$\forall x \exists y Mother(x,y)$

"Everyone has a mother" **(correct)**

$\exists y \forall x Mother(x,y)$

"There is a person who is the mother of everyone" **(wrong)**

Properties of Quantifiers

Quantifier duality

$\forall x Likes(x, IceCream)$ **is the same as** $\neg\exists x \neg Likes(x, IceCream)$

$\exists x Likes(x, Broccoli)$ **is the same as** $\neg\forall x \neg Likes(x, Broccoli)$

Fun with Sentences

- **“Brothers are siblings”**

  $\forall x, y \; (Brother(x, y) \Rightarrow Sibling(x, y))$

Fun with Sentences

- **“Brothers are siblings”**

  $\forall x, y\ (Brother(x,y) \Rightarrow Sibling(x,y))$

- **“Sibling” is symmetric**

  $\forall x, y\ (Sibling(x,y) \Leftrightarrow Sibling(y,x))$

Fun with Sentences

- **“Brothers are siblings”**

  $\forall x, y\ (Brother(x,y) \Rightarrow Sibling(x,y))$

- **“Sibling” is symmetric**

  $\forall x, y\ (Sibling(x,y) \Leftrightarrow Sibling(y,x))$

- **“One’s mother is one’s female parent”**

  $\forall x, y\ (Mother(x,y) \Leftrightarrow (Female(x) \land Parent(x,y)))$

Fun with Sentences

- **“Brothers are siblings”**

  $$\forall x, y\ (Brother(x, y) \Rightarrow Sibling(x, y))$$

- **“Sibling” is symmetric**

  $$\forall x, y\ (Sibling(x, y) \Leftrightarrow Sibling(y, x))$$

- **“One’s mother is one’s female parent”**

  $$\forall x, y\ (Mother(x, y) \Leftrightarrow (Female(x) \land Parent(x, y)))$$

- **“A first cousin is a child of a parent’s sibling”**

  $$\forall x, y\ (FirstCousin(x, y) \Leftrightarrow \exists p, ps\ (Parent(p, x) \land Sibling(ps, p) \land Parent(ps, y)))$$

Equality

Semantics

$term_1 = term_2$ **is true under a given interpretation**

if and only if

$term_1$ **and** $term_2$ **have the same interpretation**

Equality

Example

Definition of (full) sibling in terms of $Parent$

$$
\begin{aligned}
\forall x, y\, Sibling(x,y) \;\Leftrightarrow\; & (\neg(x = y) \wedge \\
& \exists m, f\ (\neg(m = f) \wedge \\
& \qquad Parent(m,x) \wedge Parent(f,x) \wedge \\
& \qquad Parent(m,y) \wedge Parent(f,y)))
\end{aligned}
$$

Properties of First-order Logic

Important notions

– validity
– satisfiability
– unsatisfiablity
– entailment

are defined for first-order logic in the same way as for propositional logic

Properties of First-order Logic

Important notions

- validity
- satisfiability
- unsatisfiablity
- entailment

are defined for first-order logic in the same way as for propositional logic

Calculi

There are sound and complete calculi for first-order logic (e.g. resolution)

- Whenever $KB \vdash \alpha$, it is also true that $KB \models \alpha$
- Whenever $KB \models \alpha$, it is also true that $KB \vdash \alpha$

But these calculi CANNOT decide validity, entailment, etc.

Properties of First-order Logic

In propositional logic

Validity, satisfiability, unsatisfiablity are decidable

In first-order logic

The set of valid, and the set of unsatisfiable formulas are enumerable

The set of satisfiable formulas is NOT EVEN enumerable