Formal Specification of Software

Reviewing Basic Set Theory

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Basic Concepts: Set

Definition

A ***set*** **is the combination of certain well-distinguished objects taken from our visual or mental experience into one entity. The objects are called the elements of the set.**

Notation

Let M **denote a set, and** m **an object. The fact that** m **is an element of** M **is denoted by**

$$m \in M.$$

Basic Concepts: Function

Definition

A *function* from a set M_1 in a set M_2 associates with an element $m_1 \in M_1$ a unique element $m_2 \in M_2$.

Notation

If f is used to denote a function this association is symbolically expressed as

$$f(m_1) = m_2.$$

The element m_1 is called the argument and m_2 the value of the function application $f(m_1)$.

Basic Concepts: Relation

Definition

A *relation* describes properties of elements, pairs of elements or in general n-tupels of elements.

Notation

If r denotes a unary relation, and a is an object, then

$$r(a)$$

denotes the fact that the relation a is true of the object a.

For a binary relation r_2 and objects a_1, a_2, the symbolic notation

$$r_2(a_1, a_1)$$

expresses that the relation r_2 is true of the pairs a_1, a_2.

Subset

Definition

A set N is called a subset of set M if every element of N is also an element of M.

N is called a superset of M in that case.

Notation

In that case, we write

$$M \subseteq N$$

Denoting Sets

- **Finite sets:** $M = \{a_1, \ldots a_n\}$
- **Reserved symbols to denote frequently occuring sets:**

  $\mathbb{N}$ **– the natural numbers**

  $\mathbb{Z}$ **– the integers**

  $\mathbb{Q}$ **– the rational numbers**

  $\mathbb{R}$ **– the real numbers**
- **Definied subsets:** $M = \{x \in N \mid \phi\}$

  (where ϕ is a property)

Examples for Defining Sets

- $\{x \in \mathbb{N} \mid x$ **is prime**$\}$

- $\{x^2 \mid x \in \mathbb{N}$ **and** x **is prime**$\}$

- $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 \mid x_i \in \mathbb{Z}\}$

Notation Related to Functions

- **An n-ary function $f : M_1 \to M_2$ is called *total* if for every n-tupel $(a_1, \ldots, a_n)$ of elements from M_1**

  $$f(a_1, \ldots, a_n) \text{ is defined.}$$

  Otherwise f is called *partial*.

- **The *range* of f is the set $\{f(a_1, \ldots, a_n) \mid a_i \in M_1\}$**

- **The *domain* of f is the set $\{m \in M_1 \mid f(m) \text{ is definied}\}$**

Operations on Sets

- **The *intersection* $A \cap B$ is the set of elements occuring in both A and B, i.e.**

$$A \cap B = \{x \mid x \in A \textbf{ and } x \in B\}$$

- **The *union* $A \cup B$ is the set of elements occuring ocurring in at least one of A, B, i.e.**

$$A \cup B = \{x \mid x \in A \textbf{ or } x \in B\}$$

- **A and B are called *disjoint* if they have no elements in common, i.e., if $A \cap B = \emptyset$**

Sets of Sets

- **The set**

$$Set(A) = \{B \mid B \subseteq A\}$$

of all subsets of A is denoted by $Set(A)$

$Set(A)$ is also called the *power set* of A

- **The set**

$$\{B \mid B \subseteq A \text{ and } B \text{ is finite}\}$$

of all finite subsets of A is denoted by $Set_\omega(A)$

- **For each natural number $n \in \mathbb{N}$, the set of all subsets of A with exactly (resp. at most) n elements is denoted by**

$$Set_n(A) \quad \text{resp.} \quad Set_{\leq n}(A).$$

Bags

Definition

A *bag* is a collection where multiple occurences of objects are possible.

Bags are sometimes also called multisets.

If B is a bag and e an arbitrary object the function, then

$count_B(e)$ denotes the number of occurences of e in B.

Examples

$\{a, b, a, c, b\}$ and $\{a, b, c\}$ are the same set,

but they are different as bags

$\{a, b, a, c, b\}$ and $\{c, b, a, b, a\}$ are identical bags.