Introduction to Artificial Intelligence

First-order Logic

(Logic, Deduction, Knowledge Representation)

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Outline

- Why first-order logic?
- Syntax and semantics of first-order logic
- Fun with sentences
- Wumpus world in first-order logic

Pros and Cons of Propositional Logic

- 😊 **Propositional logic is declarative:**
  pieces of syntax correspond to facts
- 😊 **Propositional logic allows partial / disjunctive / negated information (unlike most data structures and databases)**
- 😊 **Propositional logic is compositional:**
  meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- 😊 **Meaning in propositional logic is context-independent**
  (unlike natural language, where meaning depends on context)
- ☹ **Propositional logic has very limited expressive power**
  (unlike natural language)

  Example:
  Cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

First-order Logic

Propositional logic

Assumes that the world contains facts

First-order logic

Assumes that the world contains

- **Objects**
  people, houses, numbers, theories, Donald Duck, colors, centuries, ...
- **Relations**
  red, round, prime, multistoried, ...
  brother of, bigger than, part of, has color, occurred after, owns, ...
- **Functions**
  $+$, middle of, father of, one more than, beginning of, ...

Syntax of First-order Logic: Basic Elements

Symbols

Constants	$KingJohn,\ 2,\ Koblenz,\ C, \ldots$
Predicates	$Brother,\ >,\ =,\ \ldots$
Functions	$Sqrt,\ LeftLegOf,\ \ldots$
Variables	$x,\ y,\ a,\ b,\ \ldots$
Connectives	$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
Quantifiers	$\forall \quad \exists$

Note

The equality predicate is always in the vocabulary

It is written in infix notation

Syntax of First-order Logic: Atomic Sentences

Atomic sentence

$\textit{predicate} \; (\; \textit{term}_1, \; \ldots, \; \textit{term}_n \;)$

or

$\textit{term}_1 = \textit{term}_2$

Term

$\textit{function} \; (\; \textit{term}_1, \; \ldots, \; \textit{term}_n \;)$

or

constant

or

variable

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{\underbrace{\mathit{Brother}}_{\textbf{predicate}}\,(\,\underbrace{\underbrace{\mathit{KingJohn}}_{\textbf{constant}}}_{\textbf{term}},\ \underbrace{\underbrace{\mathit{RichardTheLionheart}}_{\textbf{constant}}}_{\textbf{term}}\,)}_{\textbf{atomic sentence}}$$

Syntax of First-order Logic: Atomic Sentences

Example

$$\underbrace{\underbrace{>}_{\text{predicate}} \left(\underbrace{\underbrace{Length}_{\text{function}}(\underbrace{LeftLegOf}_{\text{function}}(\underbrace{Richard}_{\text{constant}}))}_{\text{term}}, \underbrace{\underbrace{Length}_{\text{function}}(\underbrace{LeftLegOf}_{\text{function}}(\underbrace{KingJohn}_{\text{constant}}))}_{\text{term}} \right)}_{\text{atomic sentence}}$$

Syntax of First-order Logic: Complex Sentences

Built from atomic sentences using connectives

$$\neg S \qquad S_1 \wedge S_2 \qquad S_1 \vee S_2 \qquad S_1 \Rightarrow S_2 \qquad S_1 \Leftrightarrow S_2$$

(as in propositional logic)

Example

$$\underbrace{\underbrace{\underbrace{Sibling}_{\text{predicate}}(\underbrace{KingJohn}_{\text{term}},\underbrace{Richard}_{\text{term}})}_{\text{atomic sentence}} \Rightarrow \underbrace{\underbrace{Sibling}_{\text{predicate}}(\underbrace{Richard}_{\text{term}},\underbrace{KingJohn}_{\text{term}})}_{\text{atomic sentence}}}_{\text{complex sentence}}$$

Semantics in First-order Logic

Models of first-order logic

Sentences are true or false with respect to models, which consist of

- **a domain (also called universe)**
- **an interpretation**

Domain

A non-empty (finite or infinite) set of arbitrary elements

Interpretation

Assigns to each

- **constant symbol: a domain element**
- **predicate symbol: a relation on the domain (of appropriate arity)**
- **function symbol: a function on the domain (of appropriate arity)**

Semantics in First-order Logic

Definition

An atomic sentence

$$\textit{predicate} \; (\; \textit{term}_1, \; \ldots, \; \textit{term}_n \;)$$

is true in a certain model (that consists of a domain and an interpretation)

iff

the domain elements that are the interpretations of $term_1, \ldots, term_n$
are in the relation that is the interpretation of *predicate*

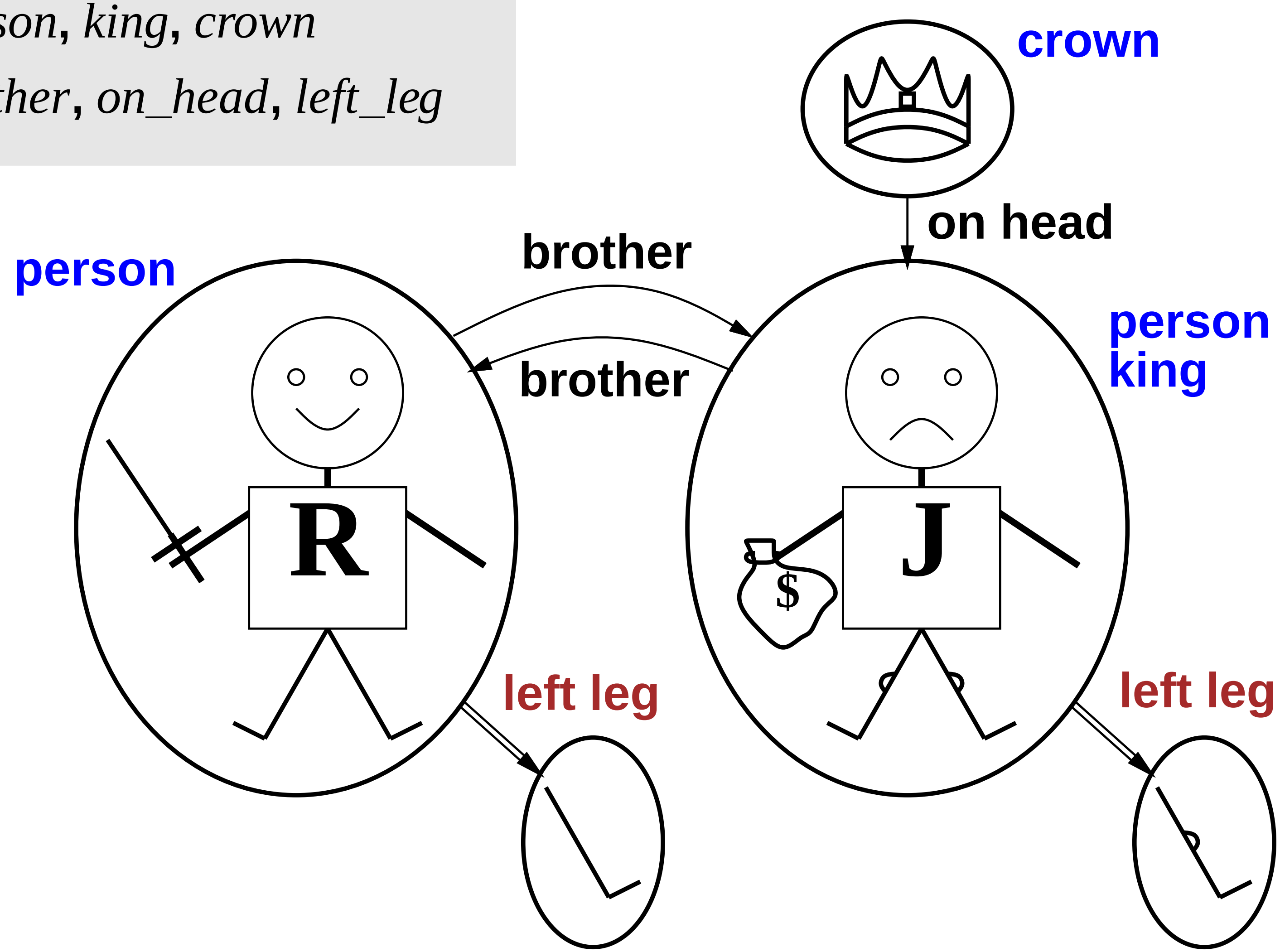
The truth value of a complex sentence in a model
is computed from the truth-values of its atomic sub-sentences
in the same way as in propositional logic

Models for First-order Logic: Example

Constants: *KingJohn*, *Richard*

Predicates: *person*, *king*, *crown*

Functions: *brother*, *on_head*, *left_leg*



Universal Quantification: Syntax

Syntax

$\forall$ *variables sentence*

Example

“Everyone studying in Koblenz is smart:

$$\forall \underbrace{x}_{\text{variables}} \underbrace{(StudiesAt(x, Koblenz) \Rightarrow Smart(x))}_{\text{sentence}}$$

Universal Quantification: Semantics

Semantics

$\forall x P$ **is true in a model**

iff

for all domain elements d **in the model:**
P **is true in the model when** x **is interpreted by** d

Intuition

$\forall x P$ **is roughly equivalent to the conjunction of all instances of** P

Example $\forall x \, StudiesAt(x, Koblenz) \Rightarrow Smart(x)$ **equivalent to:**

$$\begin{aligned} & StudiesAt(KingJohn, Koblenz) \Rightarrow Smart(KingJohn) \\ \wedge \quad & StudiesAt(Richard, Koblenz) \Rightarrow Smart(Richard) \\ \wedge \quad & StudiesAt(Koblenz, Koblenz) \Rightarrow Smart(Koblenz) \\ \wedge \quad & \ldots \end{aligned}$$

A Common Mistake to Avoid

Note

$\Rightarrow$ **is the main connective with** $\forall$

Common mistake

Using $\land$ **as the main connective with** $\forall$

Example

Correct: $\forall x\, (StudiesAt(x, Koblenz) \Rightarrow Smart(x))$

"Everyone who studies at Koblenz is smart"

Wrong: $\forall x\, (StudiesAt(x, Koblenz) \land Smart(x))$

"Everyone studies at Koblenz and is smart", i.e.,

"Everyone studies at Koblenz and everyone is smart"

Existential Quantification: Syntax

Syntax

$\exists$ *variables sentence*

Example

“Someone studying in Landau is smart:

$$\underbrace{\exists \quad x}_{\text{variables}} \quad \underbrace{(StudiesAt(x, Landau) \land Smart(x))}_{\text{sentence}}$$

Existential Quantification: Semantics

Semantics

$\exists x P$ **is true in a model**

iff

there is a domain element d **in the model such that:**
P **is true in the model when** x **is interpreted by** d

Intuition

$\exists x P$ **is roughly equivalent to the disjunction of all instances of** P

Example $\exists x \, StudiesAt(x, Landau) \land Smart(x)$ **equivalent to:**

$$\begin{aligned} & StudiesAt(KingJohn, Landau) \land Smart(KingJohn) \\ \lor \quad & StudiesAt(Richard, Landau) \land Smart(Richard) \\ \lor \quad & StudiesAt(Landau, Landau) \land Smart(Landau) \\ \lor \quad & \ldots \end{aligned}$$

Another Common Mistake to Avoid

Note

$\wedge$ **is the main connective with** $\exists$

Common mistake

Using $\Rightarrow$ **as the main connective with** $\exists$

Example

Correct: $\exists x\,(StudiesAt(x, Landau) \wedge Smart(x))$

"There is someone who studies at Landau and is smart"

Wrong: $\exists x\,(StudiesAt(x, Landau) \Rightarrow Smart(x))$

"There is someone who, if he/she studies at Landau, is smart"

This is true if there is anyone not studying at Landau

Properties of Quantifiers

Quantifiers of same type commute

$\forall x \forall y$ **is the same as** $\forall y \forall x$

$\exists x \exists y$ **is the same as** $\exists y \exists x$

Properties of Quantifiers

Quantifiers of different type do NOT commute

$\exists x \forall y$ **is not the same as** $\forall y \exists x$

Example

$\exists x \forall y Loves(x,y)$

"There is a person who loves everyone in the world"

$\forall y \exists x Loves(x,y)$

"Everyone in the world is loved by at least one person"

(Both hopefully true but different)

Properties of Quantifiers

Quantifiers of different type do NOT commute

$\exists x \forall y$ **is not the same as** $\forall y \exists x$

Example

$\forall x \exists y Mother(x,y)$

"Everyone has a mother" **(correct)**

$\exists y \forall x Mother(x,y)$

"There is a person who is the mother of everyone" **(wrong)**

Properties of Quantifiers

Quantifier duality

$\forall x Likes(x, IceCream)$ **is the same as** $\neg \exists x \neg Likes(x, IceCream)$

$\exists x Likes(x, Broccoli)$ **is the same as** $\neg \forall x \neg Likes(x, Broccoli)$

Fun with Sentences

- **“Brothers are siblings”**

  $$\forall x,y\ (Brother(x,y) \Rightarrow Sibling(x,y))$$

- **“Sibling” is symmetric**

  $$\forall x,y\ (Sibling(x,y) \Leftrightarrow Sibling(y,x))$$

- **“One’s mother is one’s female parent”**

  $$\forall x,y\ (Mother(x,y) \Leftrightarrow (Female(x) \land Parent(x,y)))$$

- **“A first cousin is a child of a parent’s sibling”**

  $$\forall x,y\ (FirstCousin(x,y) \Leftrightarrow \exists p, ps\ (Parent(p,x) \land Sibling(ps,p) \land Parent(ps,y)))$$

Equality

Semantics

$term_1 = term_2$ **is true under a given interpretation**

if and only if

$term_1$ **and** $term_2$ **have the same interpretation**

Equality

Example

Definition of (full) sibling in terms of *Parent*

$$\begin{aligned}\forall x,y\ Sibling(x,y) \Leftrightarrow\ & (\neg(x=y) \land \\ & \exists m,f\ (\neg(m=f) \land \\ & \quad Parent(m,x) \land Parent(f,x) \land \\ & \quad Parent(m,y) \land Parent(f,y)))\end{aligned}$$

Properties of First-order Logic

Important notions

- validity
- satisfiability
- unsatisfiablity
- entailment

are defined for first-order logic in the same way as for propositional logic

Calculi

There are sound and complete calculi for first-order logic (e.g. resolution)

- **Whenever** $KB \vdash \alpha$**, it is also true that** $KB \models \alpha$
- **Whenever** $KB \models \alpha$**, it is also true that** $KB \vdash \alpha$

But these calculi CANNOT decide validity, entailment, etc.

Properties of First-order Logic

In propositional logic

Validity, satisfiability, unsatisfiablity are decidable

In first-order logic

The set of valid, and the set of unsatisfiable formulas are enumerable

The set of satisfiable formulas is NOT EVEN enumerable