Logical Foundations of Cyber-Physical Systems

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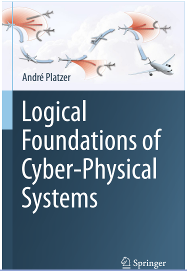
http://lfcps.org/



Outline

Outline (Introduction to CPS)

1. CPS are Multi-Dynamical Systems
2. CPS Programs
   - Syntax
   - Semantics
   - Examples
3. Differential Dynamic Logic
   - Syntax
   - Semantics
   - Example: Car Control Design
4. Dynamic Axioms for Dynamical Systems
   - Axiomatics
   - dL Proofs in KeYmaera X
5. Differential Invariants for Differential Equations
   - Axiomatics
   - Examples
6. Summary

Cyber-Physical Systems Analysis: Aircraft Example

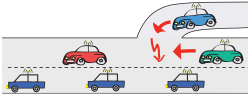
Which control decisions are safe for aircraft collision avoidance?

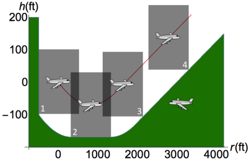
Cyber-Physical Systems

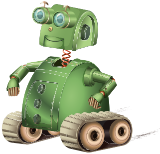
CPSs combine cyber capabilities with physical capabilities
to solve problems that neither part could solve alone.

Prospects: Safety & Efficiency

(Autonomous) cars | (Auto)Pilot support | Robots near humans







Cyber-Physical Systems

CPSs combine cyber capabilities with physical capabilities
to solve problems that neither part could solve alone.

Can you trust a computer to control physics?

Can you trust a computer to control physics?

1. Depends on how it has been programmed
2. And on what will happen if it malfunctions

Rationale

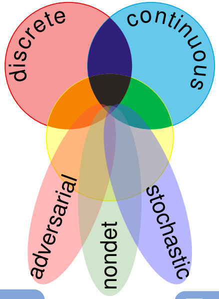
1. Safety guarantees require analytic foundations.
2. A common foundational core helps all application domains.
3. Foundations revolutionized digital computer science & our society.
4. Need even stronger foundations when software reaches out into our physical world.

CPSs deserve proofs as safety evidence!

CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

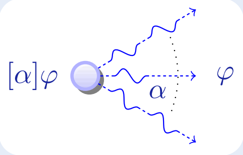
Descriptive simplification

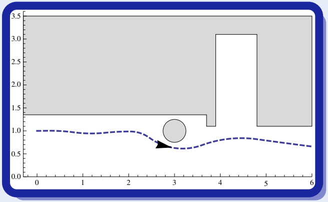
Tame Parts

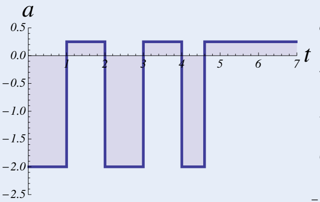
Exploiting compositionality tames CPS complexity.

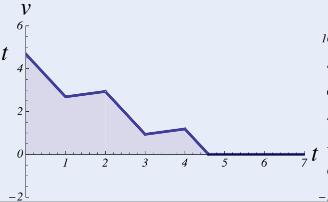
Analytic simplification

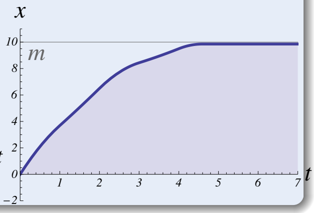
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)



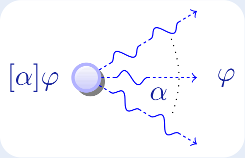


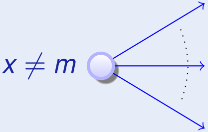


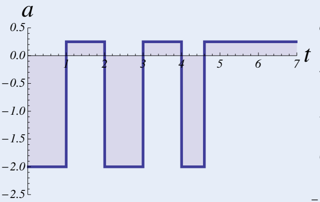


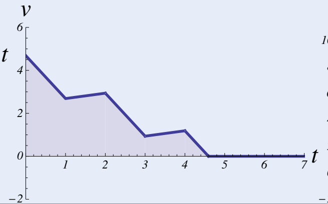


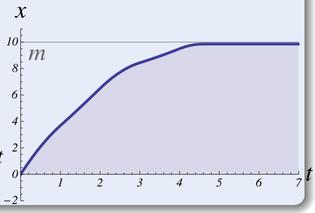
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)











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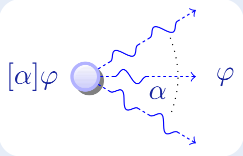

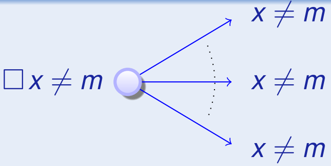

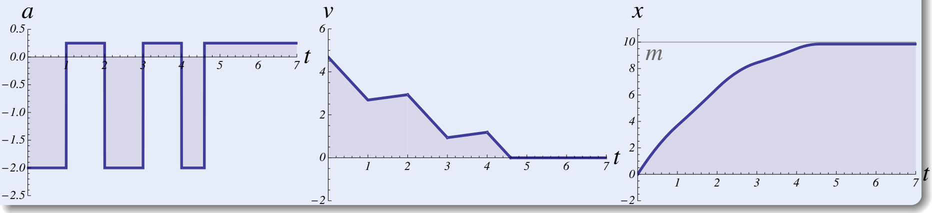

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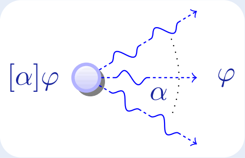

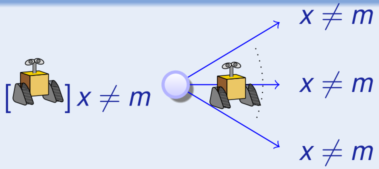

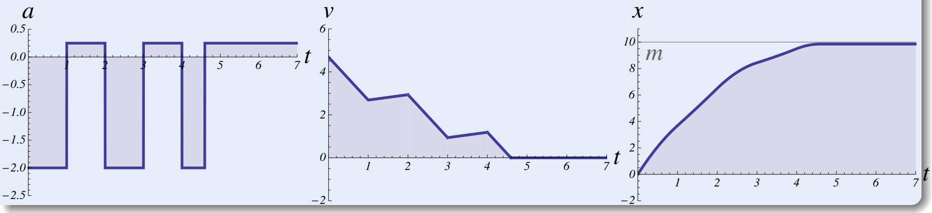

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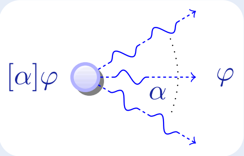
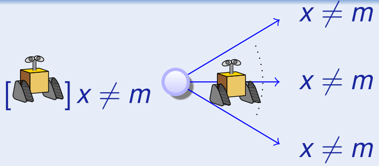
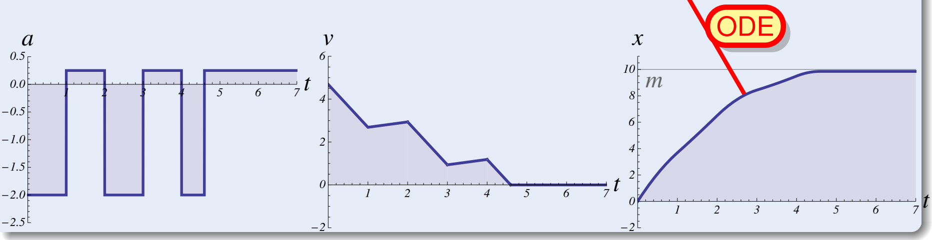




$x' = v, v' = a$



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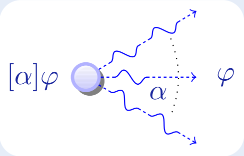
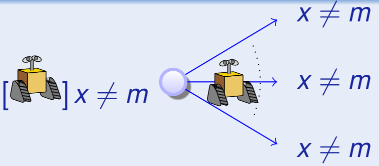
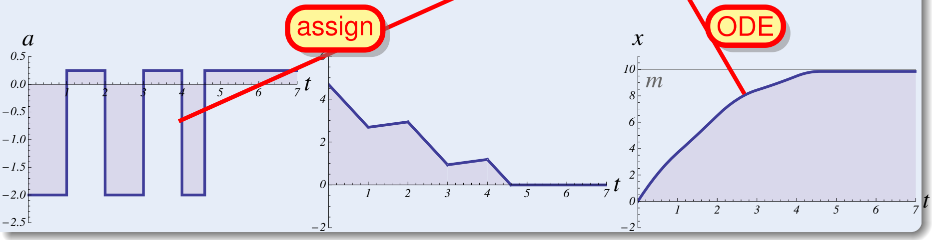




$a := -b$ $\quad x' = v, v' = a$



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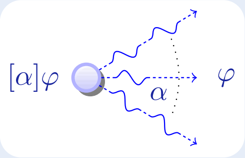
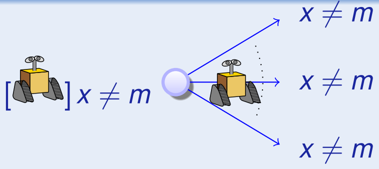
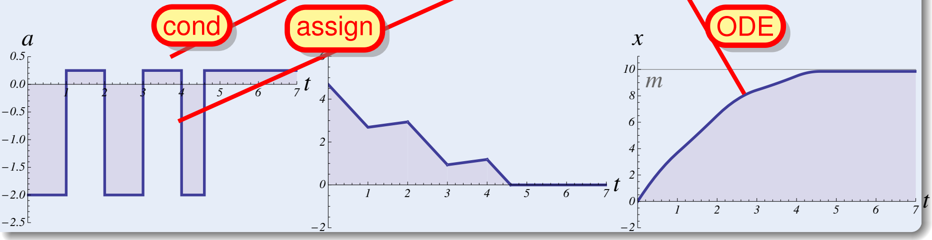




$(\text{if}(SB(x,m))\ a := -b)\quad x' = v, v' = a$



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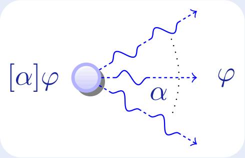
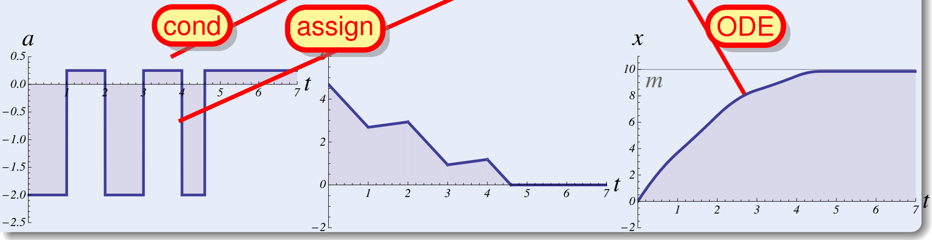


seq. compose

$$(\text{if}(SB(x,m))\ a := -b)\ ;\ x' = v, v' = a$$



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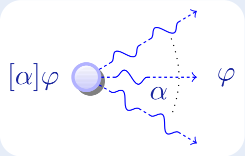


seq. compose

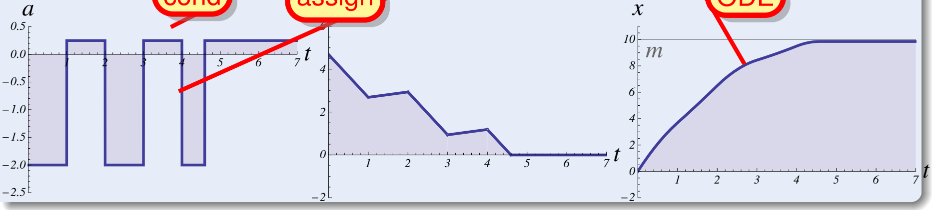
nondet. repeat

$$\big((\text{if}(SB(x,m))\ a := -b)\ ;\ x' = v, v' = a\big)^*$$

cond

assign

ODE



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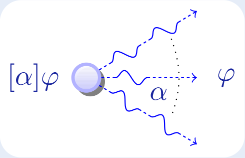
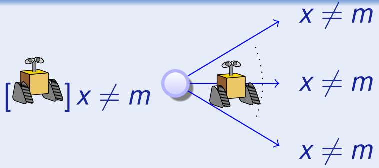
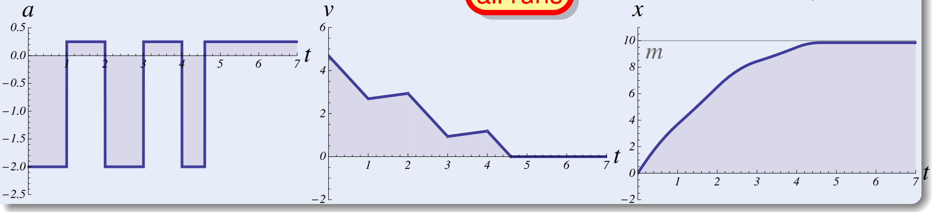




$$[((\text{if}(SB(x,m))\ \ a := -b)\ ;\ x' = v, v' = a)^*]\underbrace{x \neq m}_{\text{post}}$$

all runs



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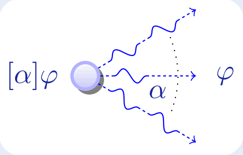
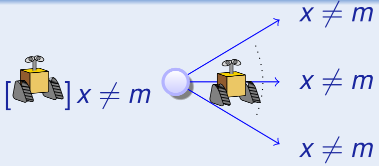
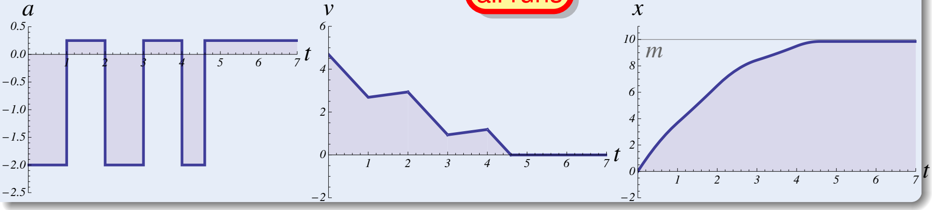




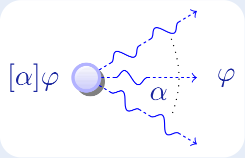
$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \Big[\Big(\big(\text{if}(\text{SB}(x,m)) \quad a := -b\big) \;;\; x' = v, v' = a\Big)^*\Big] \underbrace{x \neq m}_{\text{post}}$$

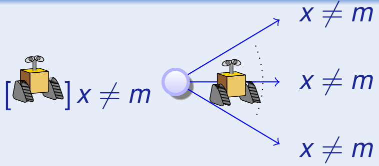
all runs



CPS Analysis

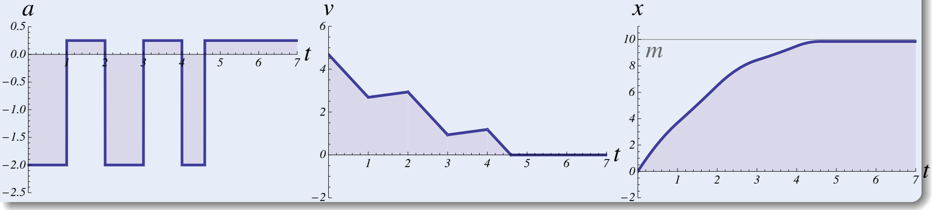
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)





nondet. choice

$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \left[\left(\left(?\neg SB(x,m) \cup a := -b\right) ;\; x' = v, v' = a\right)^*\right] \underbrace{x \neq m}_{\text{post}}$$



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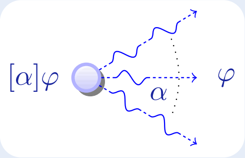
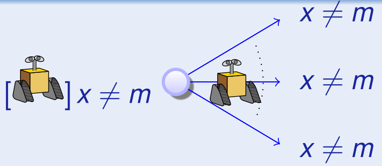
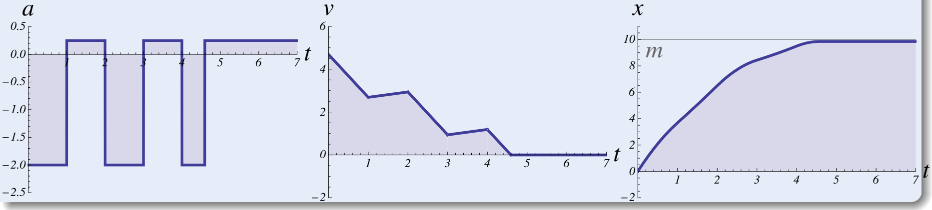




test

nondet. choice

$$\underbrace{x \neq m \land b > 0}_{\text{init}} \rightarrow \left[\left((?\neg SB(x,m) \cup a := -b) ; x' = v, v' = a\right)^*\right]\underbrace{x \neq m}_{\text{post}}$$



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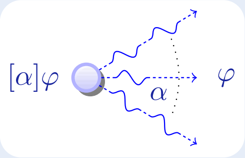

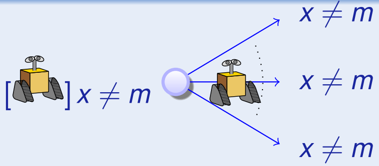

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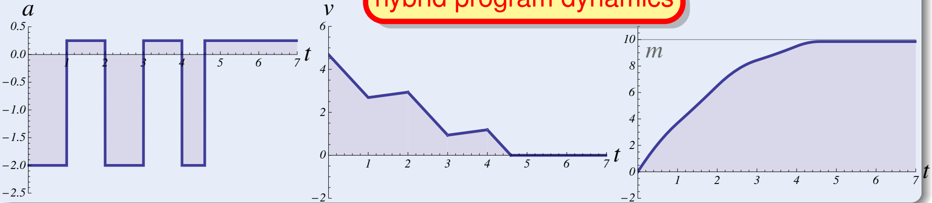
hybrid program dynamics



Outline (Modeling CPS)

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

- Discrete Assign
- Test Condition
- Differential Equation
- Nondet. Choice
- Seq. Compose
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Hybrid Programs: Syntax

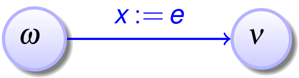
Definition (Syntax of hybrid program α)

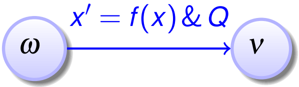
$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

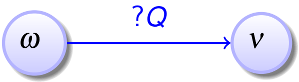
Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Like regular expressions. Everything nondeterministic

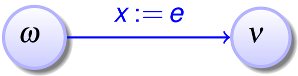
Hybrid Programs: Semantics

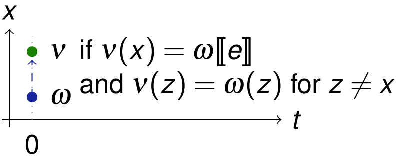


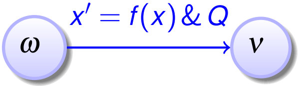


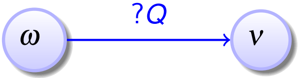


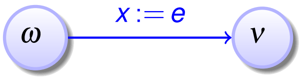
Hybrid Programs: Semantics

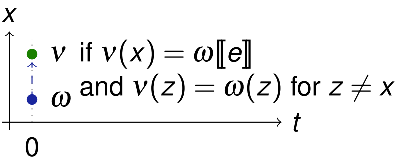


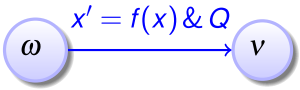


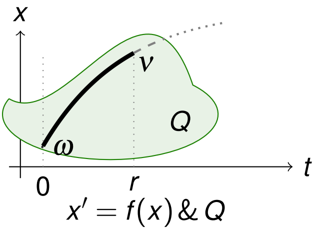


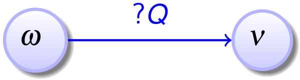



ω
x := e
ν

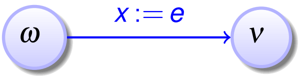

x
ν if ν(x) = ω⟦e⟧
ω and ν(z) = ω(z) for z ≠ x
t
0

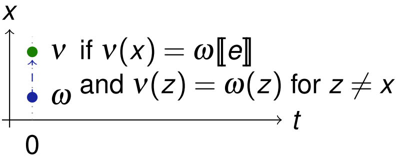

ω
x′ = f(x) & Q
ν

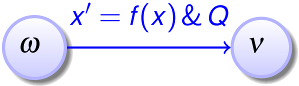

x
ν
Q
ω
t
0
r
x′ = f(x) & Q

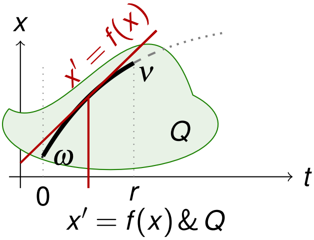

ω
?Q
ν

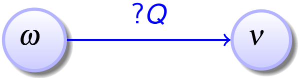
Hybrid Programs: Semantics



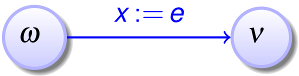


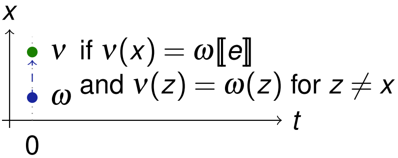


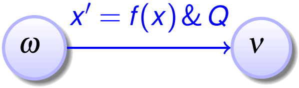


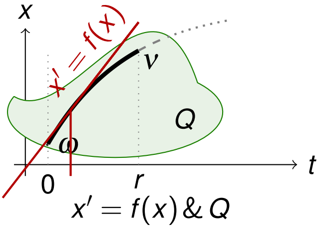


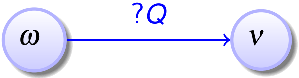
Hybrid Programs: Semantics

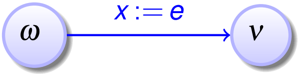


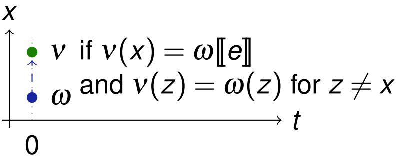


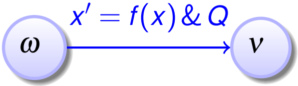


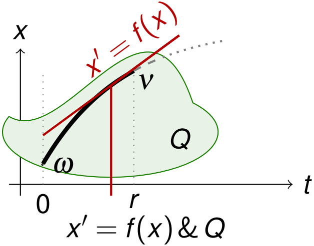


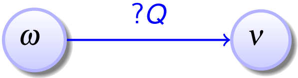



ω
x := e
ν

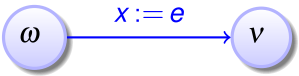

x
ν if ν(x) = ω⟦e⟧
ω and ν(z) = ω(z) for z ≠ x
t
0

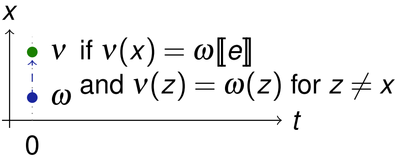

ω
x′ = f(x) & Q
ν

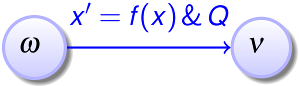

x
x′ = f(x)
ν
Q
ω
t
0
r
x′ = f(x) & Q

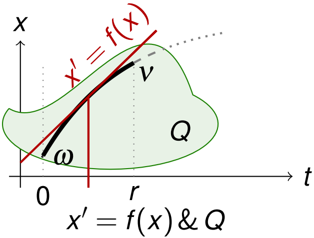

ω
?Q
ν

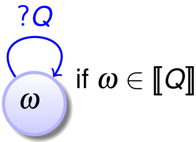
Hybrid Programs: Semantics

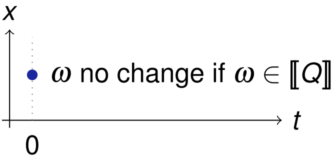


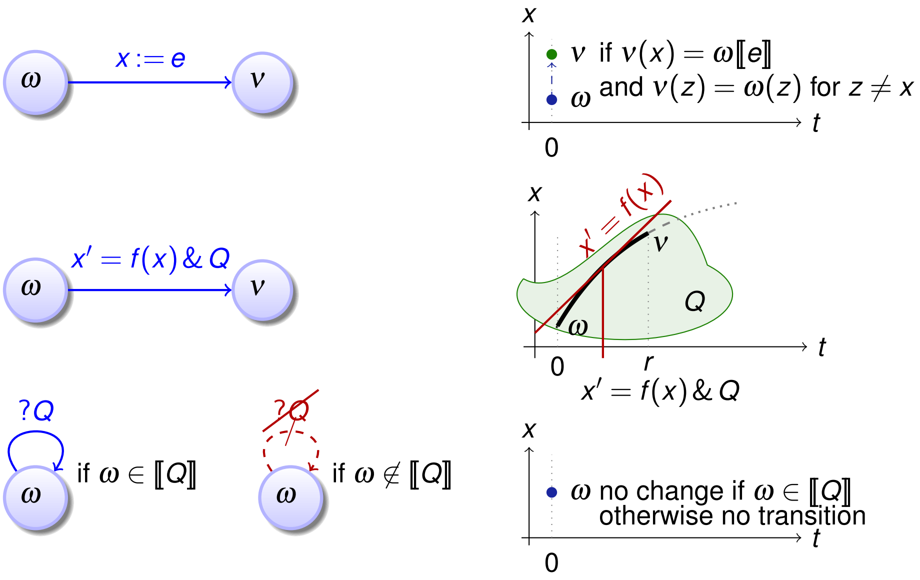






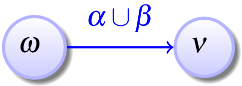


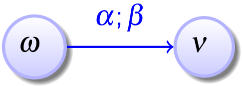


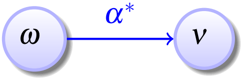


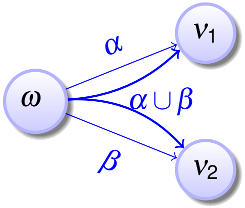
ω
$x := e$
ν
ν if $\nu(x) = \omega[\![e]\!]$
ω and $\nu(z) = \omega(z)$ for $z \neq x$
x
t
0
ω
$x' = f(x) \,\&\, Q$
ν
$x' = f(x)$
Q
r
$x' = f(x) \,\&\, Q$
?Q
if $\omega \in [\![Q]\!]$
?Q
if $\omega \notin [\![Q]\!]$
ω no change if $\omega \in [\![Q]\!]$
otherwise no transition

Hybrid Programs: Semantics

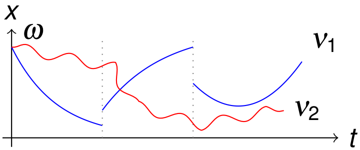




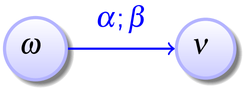




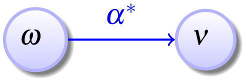
ω
v_1
v_2
α
$\alpha \cup \beta$
β



x
ω
v_1
v_2
t

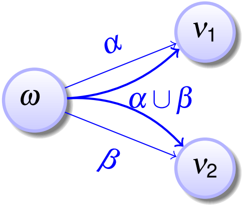


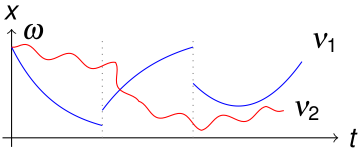
ω
$\alpha;\beta$
v

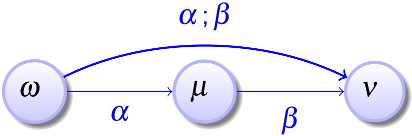


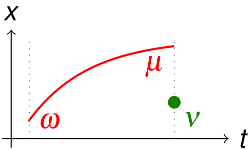
ω
α^*
v

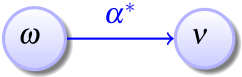
Hybrid Programs: Semantics



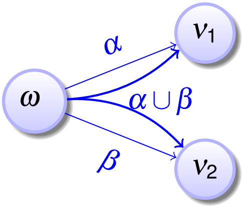


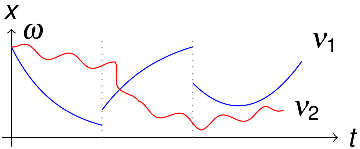


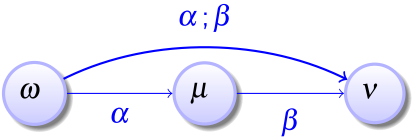


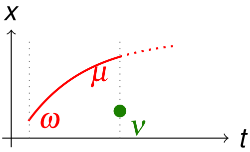


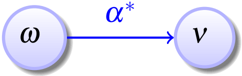
Hybrid Programs: Semantics



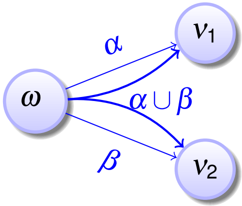


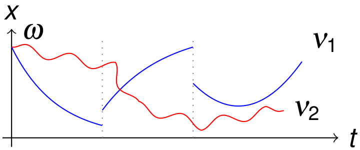


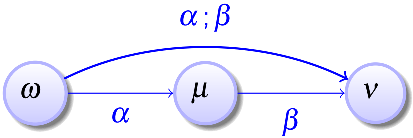


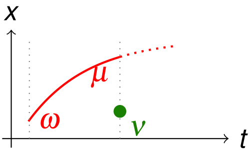


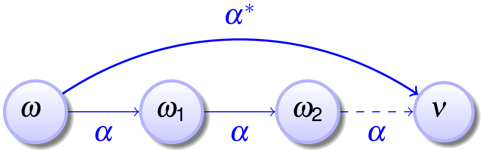
Hybrid Programs: Semantics

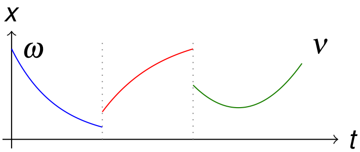




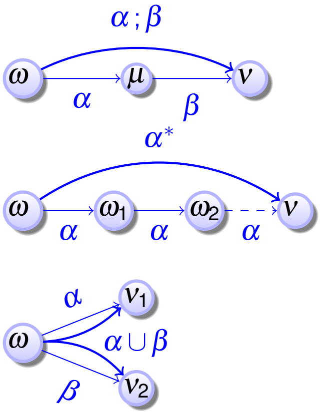




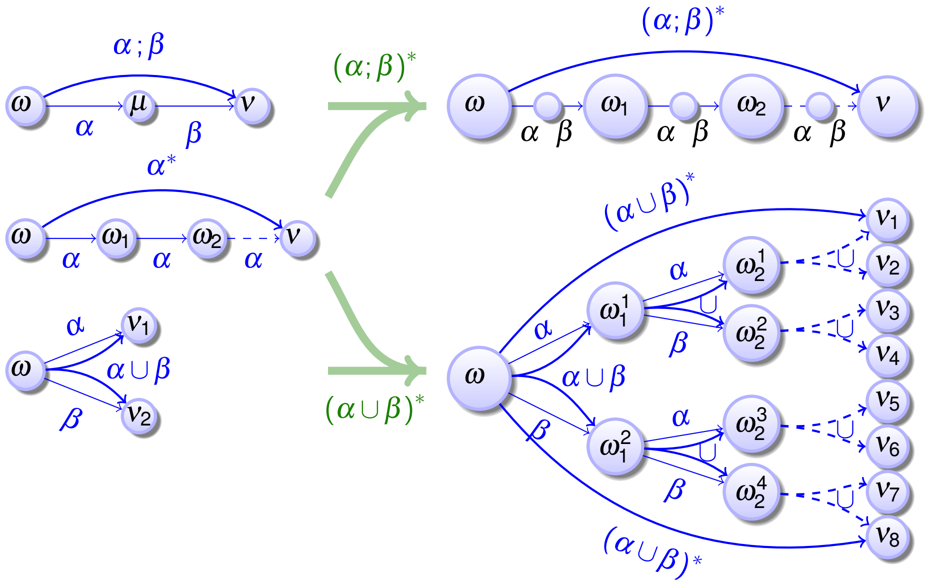




Plug-in for Semantics of Composed Hybrid Programs



Plug-in for Semantics of Composed Hybrid Programs



Hybrid Programs: Syntax & Semantics

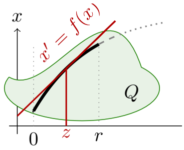
Definition (Syntax of hybrid program α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) ($[\![\cdot]\!] : \mathrm{HP} \to \wp(\mathscr{S} \times \mathscr{S})$)

$$\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) : \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] &= \{(\omega, \omega) : \omega \in [\![Q]\!]\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) : \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha; \beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] = \{(\omega, \nu) : (\omega, \mu) \in [\![\alpha]\!] \text{ and } (\mu, \nu) \in [\![\beta]\!]\} \\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!] \qquad \alpha^n \equiv \underbrace{\alpha; \alpha; \alpha; \ldots; \alpha}_{n \text{ times}}
\end{aligned}$$

compositional



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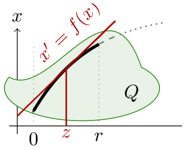
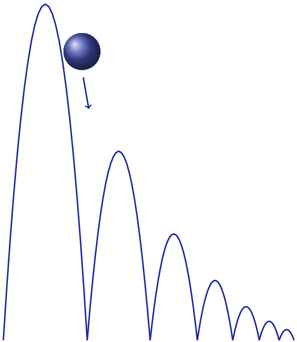
Definition (Semantics of hybrid programs) $([\![\cdot]\!] : \text{HP} \to \wp(\mathscr{S} \times \mathscr{S}))$

$$
\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] &= \{(\omega, \omega) \ : \ \omega \in [\![Q]\!]\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha ; \beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] \\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!]
\end{aligned}
$$

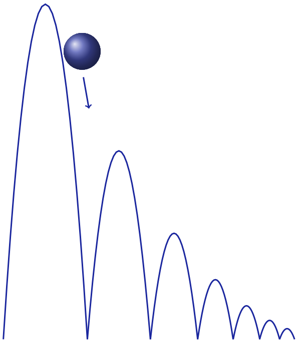
compositional

1. $\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \in [\![x' = f(x) \wedge Q]\!]$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'





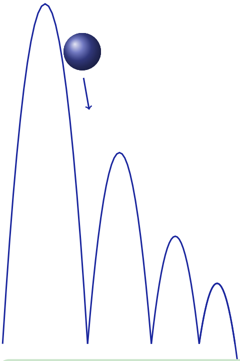
Example (Quantum the Bouncing Ball)



Example (Quantum the Bouncing Ball)

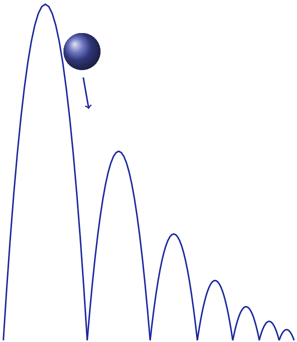
$$\{x' = v, v' = -g\}$$

Quantum the Acrophobic Bouncing Ball



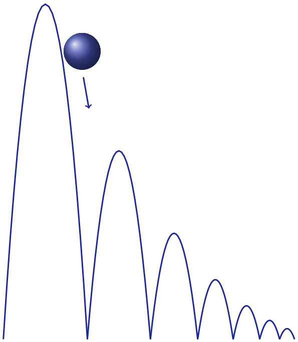
Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g\}$$



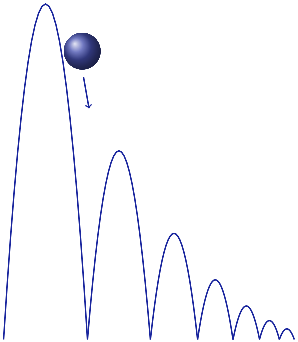
Example (Quantum the Bouncing Ball)

$$\{x' = v, v' = -g \,\&\, x \geq 0\}$$



Example (Quantum the Bouncing Ball)

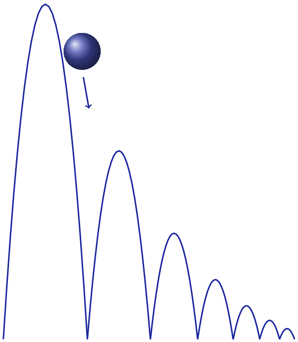
$$\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\ \ v := -cv$$



Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\;\; v := -cv\big)^*$$

Quantum Discovered a Crack in the Fabric of Time



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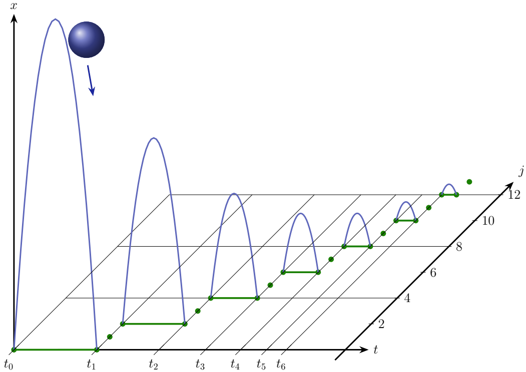

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$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
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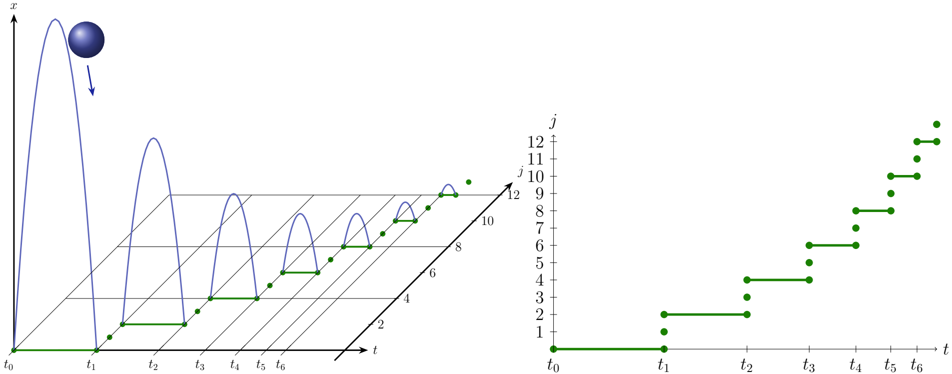
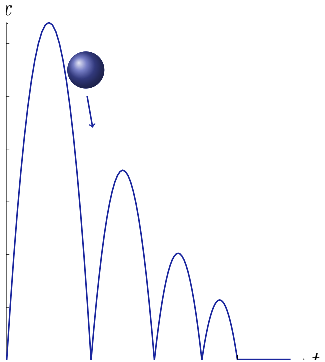
Quantum Discovered a Crack in the Fabric of Time



Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\ \ v := -cv\big)^*$$

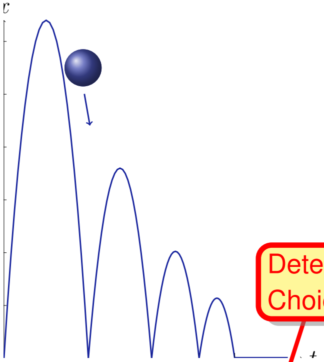
Quantum Learns to Deflate



$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv$

Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\;\; v := -cv\big)^*$$



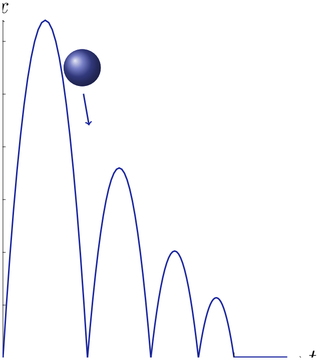
$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

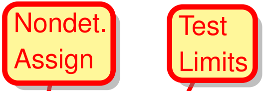
Determ. Choice

Nondet. Choice

Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\,(v := -cv \cup v := 0)\big)^*$$





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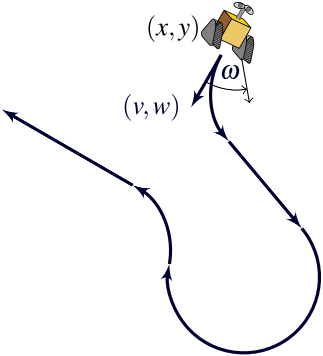
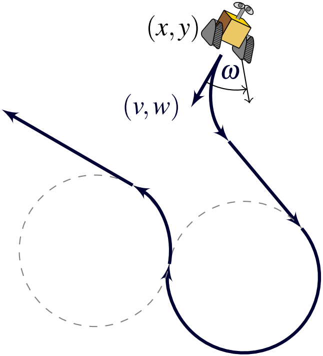
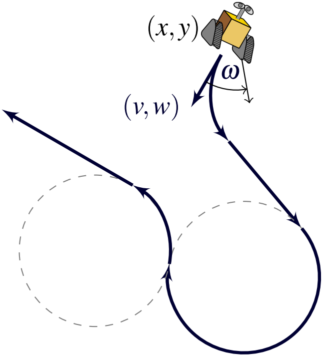

(x, y)
ω
(v, w)



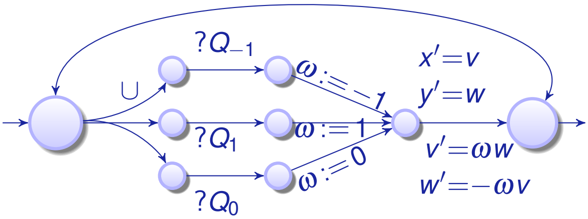
Example (Runaround Robot)

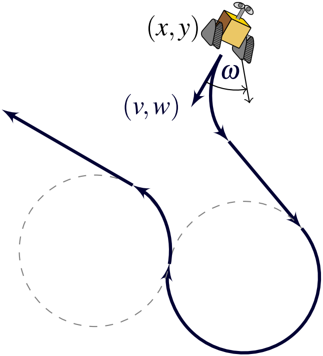
$$\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$



Example (Runaround Robot)

$$\big((?Q_{-1};\omega:=-1 \cup ?Q_1;\omega:=1 \cup ?Q_0;\omega:=0);$$
$$\{x'=v, y'=w, v'=\omega w, w'=-\omega v\}\big)^*$$





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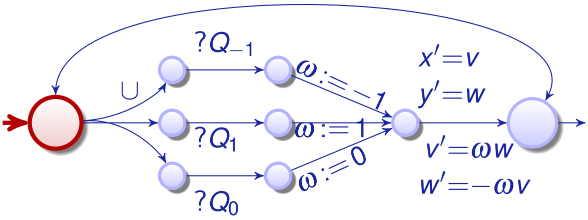

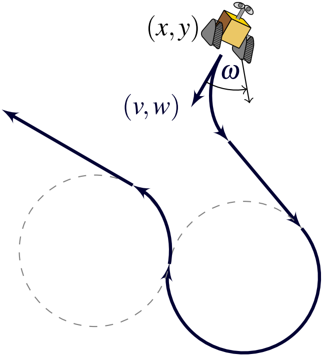

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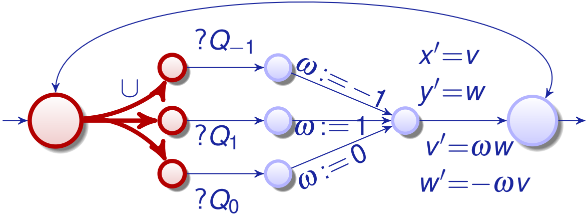

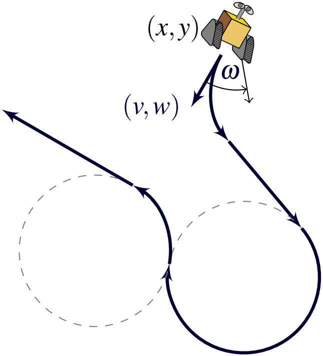

Example (Runaround Robot)

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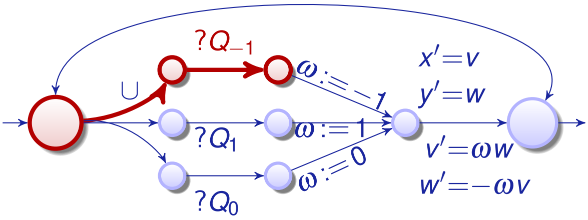
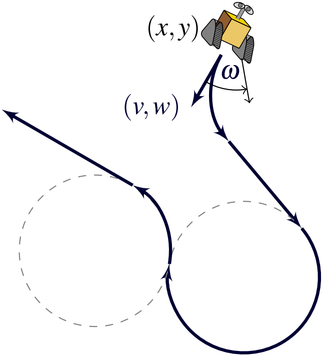
$$\left. \{x' = v, y' = w, v' = \omega w, w' = -\omega v\} \right)^*$$

Runaround Robot





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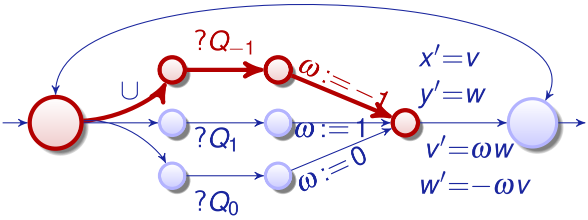

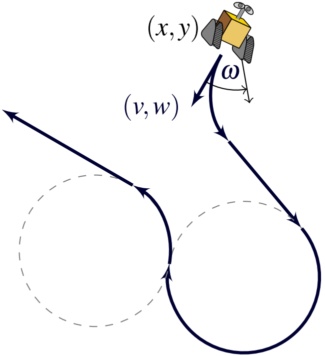

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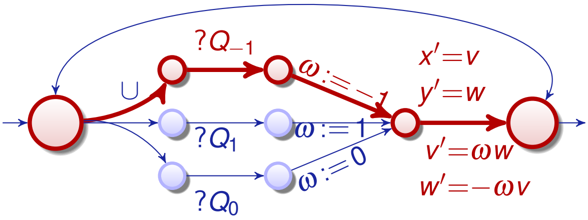

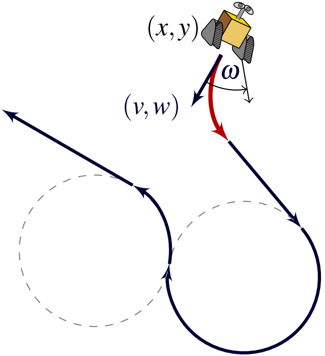

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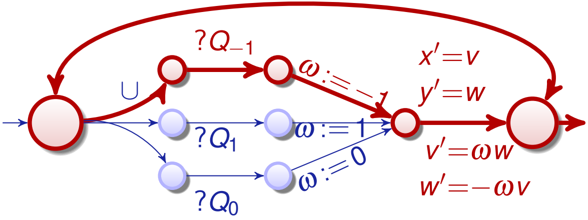

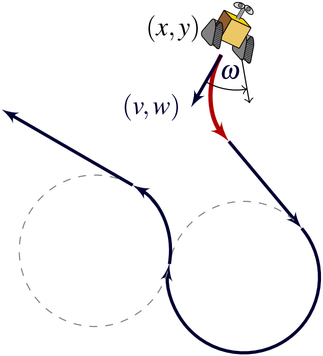

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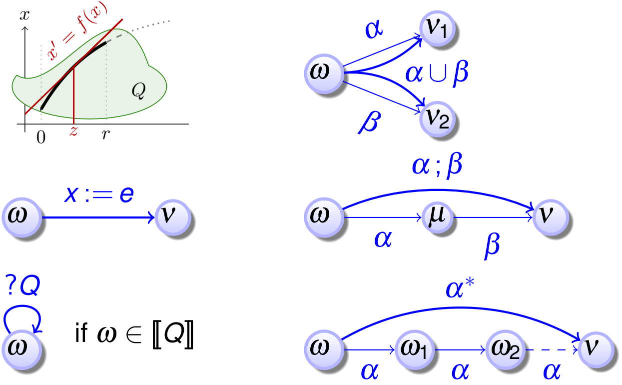
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$

Definition (Hybrid program)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid {\color{red} x' = f(x) \,\&\, Q} \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

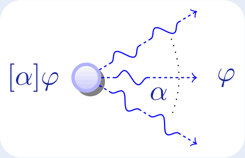


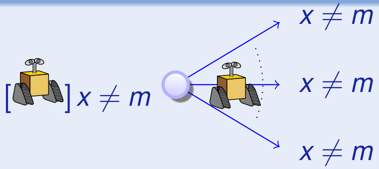
Programming CPS $\neq$ program cyber $\|$ program physics (mutual ignorance)

Outline (Specifying CPS)

1. CPS are Multi-Dynamical Systems
2. CPS Programs
   - Syntax
   - Semantics
   - Examples
3. Differential Dynamic Logic
   - Syntax
   - Semantics
   - Example: Car Control Design
4. Dynamic Axioms for Dynamical Systems
   - Axiomatics
   - dL Proofs in KeYmaera X
5. Differential Invariants for Differential Equations
   - Axiomatics
   - Examples
6. Summary

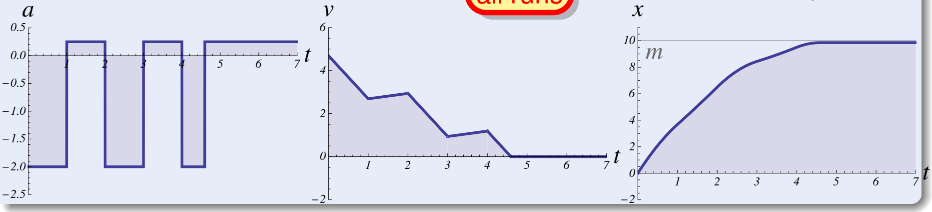
Concept (Differential Dynamic Logic) (JAR'08,LICS'12)





$$\underbrace{x \neq m \wedge b > 0}_{\text{init}} \rightarrow \Big[\big((\text{if}(SB(x,m))\quad a := -b) \,;\, x' = v, v' = a\big)^*\Big] \underbrace{x \neq m}_{\text{post}}$$

all runs



Definition (Syntax of differential dynamic logic)

The *formulas* of *differential dynamic logic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \land Q \mid P \lor Q \mid P \to Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

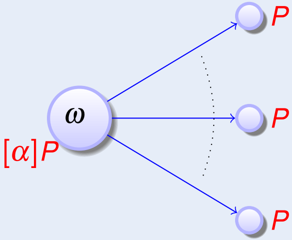
Definition (Syntax of differential dynamic logic)

The *formulas* of *differential dynamic logic* are defined by the grammar:

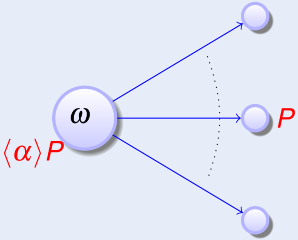
$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Not | And | Or | Imply | All reals | Some real | All runs | Some runs

Definition (dL Formulas)



Definition (dL Formulas)



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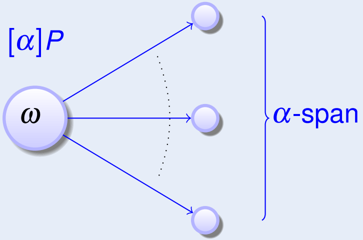
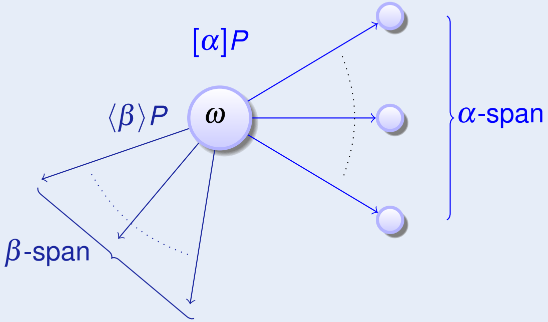
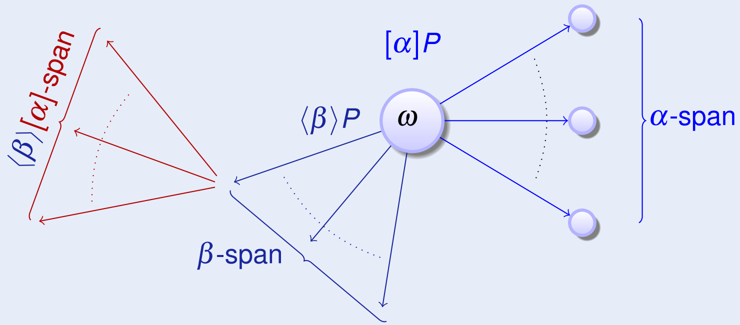


Definition (dL Formulas)



Differential Dynamic Logic dL: Semantics

Definition (dL Formulas)



Differential Dynamic Logic: Syntax & Semantics

Definition (Syntax of differential dynamic logic)

The *formulas* of *differential dynamic logic* are defined by the grammar:

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

Definition (dL semantics) $([\![\cdot]\!] : \text{Fml} \rightarrow \wp(\mathscr{S}))$

$$\begin{aligned}
[\![e \geq \tilde{e}]\!] &= \{\omega : \omega[\![e]\!] \geq \omega[\![\tilde{e}]\!]\}\\
[\![\neg P]\!] &= [\![P]\!]^{\complement} = \mathscr{S} \setminus [\![P]\!]\\
[\![P \wedge Q]\!] &= [\![P]\!] \cap [\![Q]\!]\\
[\![P \vee Q]\!] &= [\![P]\!] \cup [\![Q]\!]\\
[\![P \rightarrow Q]\!] &= [\![P]\!]^{\complement} \cup [\![Q]\!]\\
[\![\langle\alpha\rangle P]\!] &= [\![\alpha]\!] \circ [\![P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for some } \nu : (\omega,\nu) \in [\![\alpha]\!]\}\\
[\![[\alpha]P]\!] &= [\![\neg\langle\alpha\rangle\neg P]\!] = \{\omega : \nu \in [\![P]\!] \text{ for all } \nu : (\omega,\nu) \in [\![\alpha]\!]\}\\
[\![\exists x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for some } r \in \mathbb{R}\}\\
[\![\forall x\, P]\!] &= \{\omega : \omega_x^r \in [\![P]\!] \text{ for all } r \in \mathbb{R}\}
\end{aligned}$$

$$\omega_x^d(y) = \begin{cases} d & \text{if } y = x\\ \omega(y) & \text{if } y \neq x \end{cases}$$

$[\![P]\!]$ the set of states in which formula P is true

$\omega \models P$ formula P is true in state ω, alias $\omega \in [\![P]\!]$

$\models P$ formula P is valid, i.e., true in all states ω, i.e., $[\![P]\!] = \mathscr{S}$

Definition (dL semantics) $([\![\cdot]\!] : \mathrm{Fml} \to \wp(\mathscr{S}))$

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$\exists d\,[x := 1; x' = d]x \geq 0$ and $[x := x+1; x' = d]x \geq 0$ and $\langle x' = d \rangle x \geq 0$

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$\models \exists d\, [x := 1; x'=d]x \geq 0$ and $\not\models [x := x+1; x'=d]x \geq 0$ and $\not\models \langle x'=d \rangle x \geq 0$

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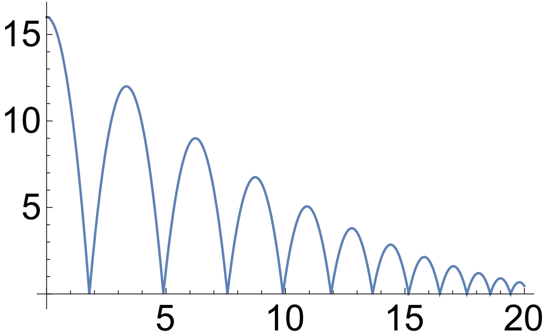

Example (Bouncing Ball)

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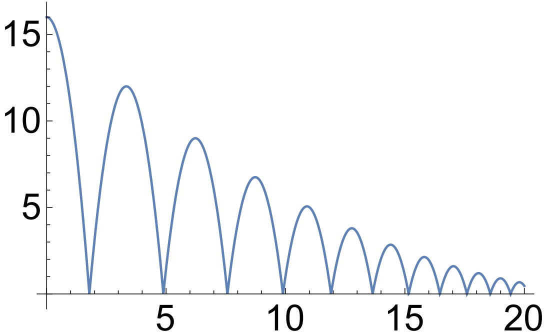
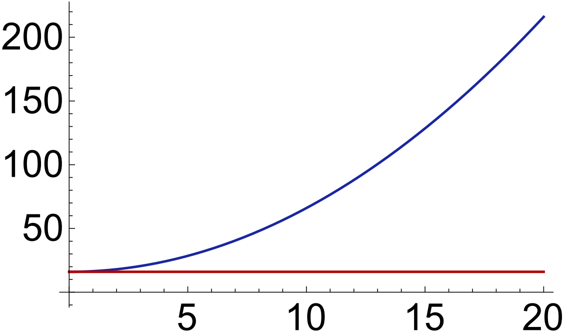


Example (Bouncing Ball)

$$H{=}x{\geq}0 \quad \rightarrow [\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*]\, 0{\leq}x{\leq}H$$

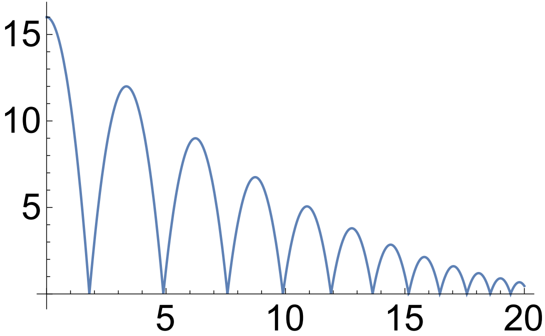
Quantum the Acrophobic Bouncing Ball



Not if $g < 0$ in anti-gravity

Example (Bouncing Ball)

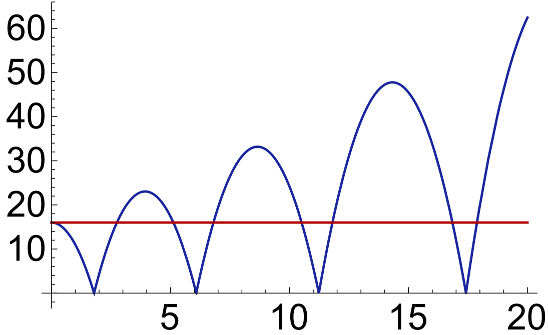
$$H{=}x{\geq}0 \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \text{if}(x = 0)\, v := -cv)^*]\, 0{\leq}x{\leq}H$$



Example (Bouncing Ball)

$$H = x \geq 0 \land g > 0 \to \left[\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\right.\right.$$
$$\left.\left.\text{if}(x = 0)\, v := -cv\right)^*\right] 0 \leq x \leq H$$

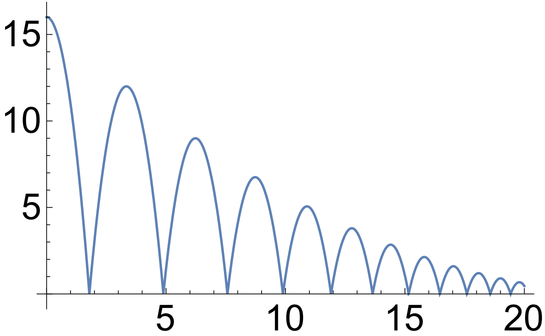
Quantum the Acrophobic Bouncing Ball



Not if $c > 1$ for anti-damping

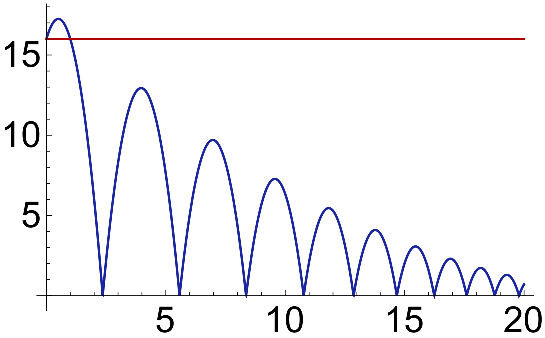
Example (Bouncing Ball)

$$H = x \geq 0 \land g > 0 \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv)^*]\, 0 \leq x \leq H$$



Example (Bouncing Ball)

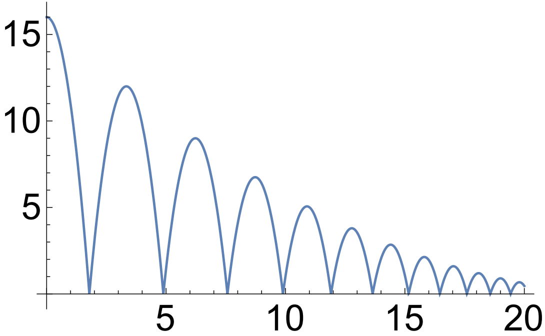
$$1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow \big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\, v := -cv\big)^*\big]\, 0 \leq x \leq H$$



Not if $v > 0$ initial climbing

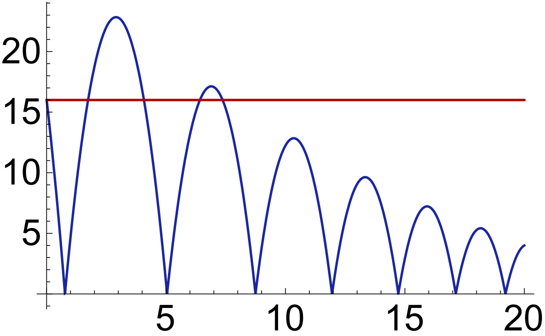
Example (Bouncing Ball)

$$1 \geq c \geq 0 \land H = x \geq 0 \land g > 0 \rightarrow [(\{x' = v, v' = -g \,\&\, x \geq 0\}; \\ \text{if}(x = 0)\, v := -cv)^*]\, 0 \leq x \leq H$$



Example (Bouncing Ball)

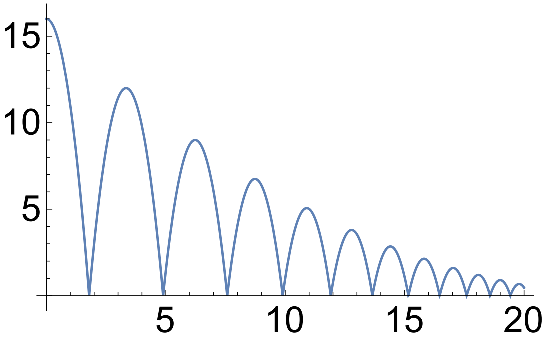
$$\color{red}{v\leq 0} \land 1\geq c\geq 0 \land H=x\geq 0 \land g>0 \rightarrow \big[\big(\{x'=v, v'=-g \,\&\, x\geq 0\};$$
$$\text{if}(x=0)\, v:=-cv\big)^*\big]\, 0\leq x\leq H$$



Not if $v \ll 0$ initial dribbling

Example (Bouncing Ball)

$$v \le 0 \land 1 \ge c \ge 0 \land H = x \ge 0 \land g > 0 \to \big[\big(\{x' = v, v' = -g \,\&\, x \ge 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*\big]\, 0 \le x \le H$$



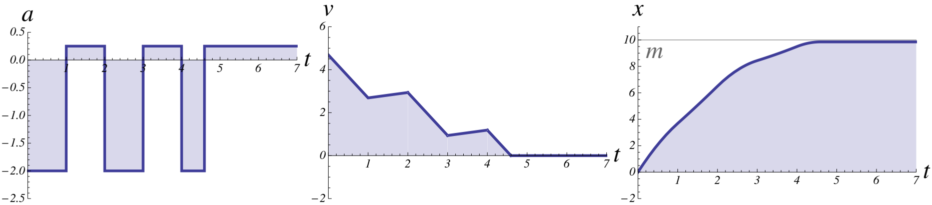
Example (Bouncing Ball)

$$v=0 \land 1\geq c\geq 0 \land H=x\geq 0 \land g>0 \rightarrow \left[\left(\{x'=v, v'=-g \,\&\, x\geq 0\};\ \text{if}(x=0)\, v:=-cv\right)^*\right] 0\leq x\leq H$$

Acceleration condition $?Q$

Example (Single car car_s)

$$\Big(((?Q;\, a:=A)\cup a:=-b);\ \{x'=v, v'=a\,\&\, v\geq 0\}\Big)^*$$



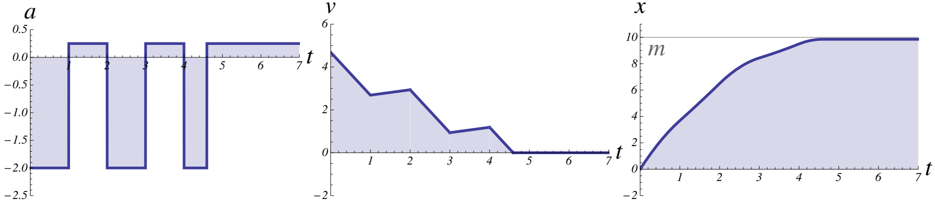
$Q \equiv$

Example (Single car car_ε time-triggered)

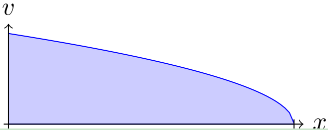
$$\Big(\big((?Q; a := A) \cup a := -b\big); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}\Big)^*$$

Example (Safely stays before traffic light m)

$$A \geq 0 \land b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$



$Q \equiv$

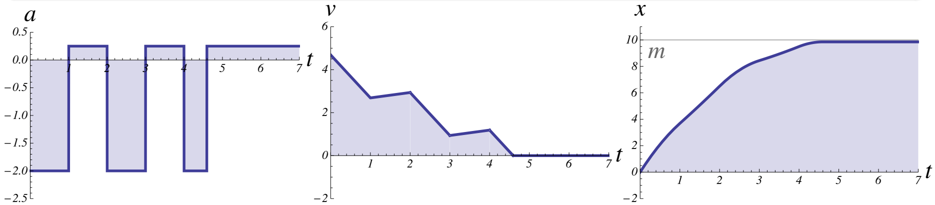


Example (Single car car_ε time-triggered)

$$\big(((?Q; a := A) \cup a := -b); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}\big)^*$$

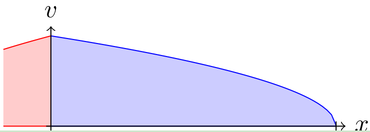
Example (▶ Safely stays before traffic light m)

$$v^2 \leq 2b(m - x) \wedge A \geq 0 \wedge b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$



Ex: Car Control Properties time-triggered

$Q \equiv 2b(m-x) \geq v^2 + \left(A+b\right)\left(A\varepsilon^2 + 2\varepsilon v\right)$

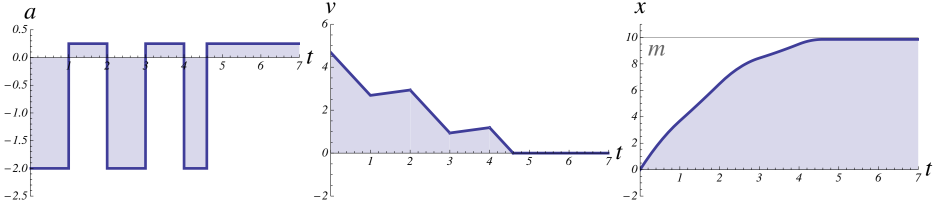


Example (Single car car_ε time-triggered)

$$\left(\left(\left(?Q; a := A\right) \cup a := -b\right); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}\right)^*$$

Example (Safely stays before traffic light m)

$$v^2 \leq 2b(m-x) \land A \geq 0 \land b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$



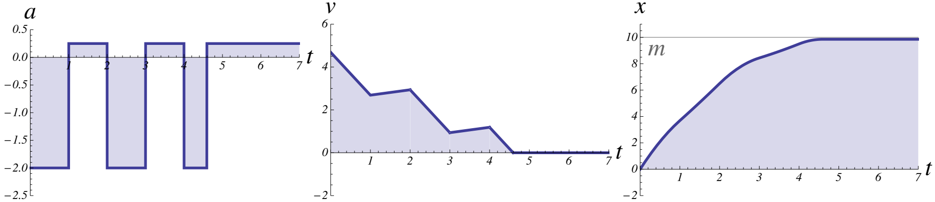
$Q \equiv 2b(m-x) \geq v^2 + (A+b)(A\varepsilon^2 + 2\varepsilon v)$

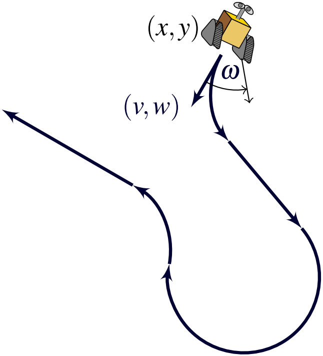
Example (Single car car_ε time-triggered)

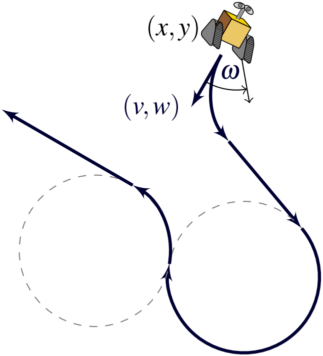
$$\big(((?Q; a := A) \cup a := -b); t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}\big)^*$$

Example (Live, can move everywhere)

$$\varepsilon > 0 \land A > 0 \land b > 0 \rightarrow \forall p \exists m \langle car_\varepsilon \rangle\, x \geq p$$

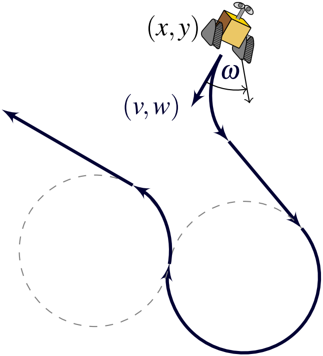



(x, y)
ω
(v, w)



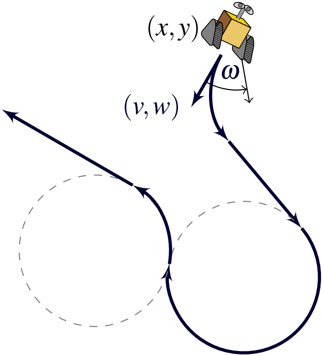
Example (Runaround Robot)

$$\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$



Example (Runaround Robot)

$$(x,y) \neq o \rightarrow \Big[\Big(\big(\omega := -1 \cup \omega := 1 \cup \omega := 0\big);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\Big)^*\Big]\,(x,y) \neq o$$



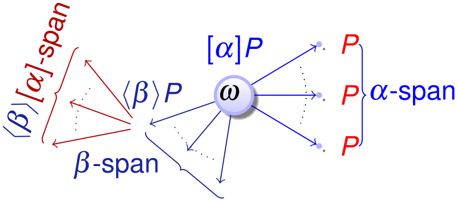
Example (Runaround Robot)

$$(x,y) \neq o \rightarrow \Big[\Big(\big(?Q_{-1};\, \omega := -1 \cup ?Q_1;\, \omega := 1 \cup ?Q_0;\, \omega := 0\big);$$
$$\{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\Big)^*\Big]\,(x,y) \neq o$$

Summary: Specifying CPS

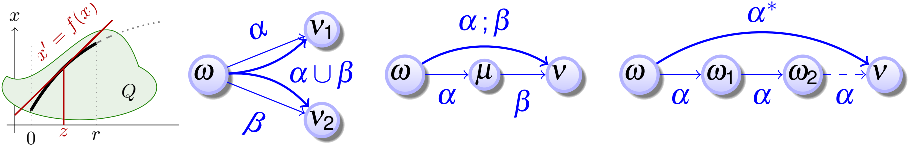
Definition (Differential dynamic logic)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$



Definition (Hybrid program)

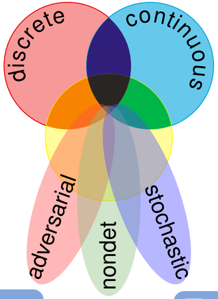
$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x)\,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$



Outline (Proving CPS)

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

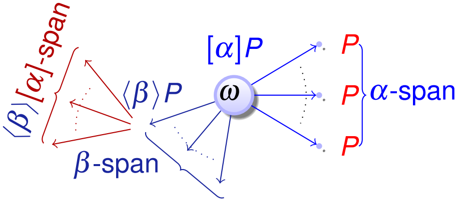
Exploiting compositionality tames CPS complexity.

Analytic simplification

Summary: Specifying CPS

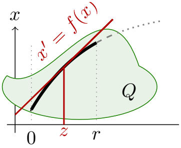
Definition (Differential dynamic logic)

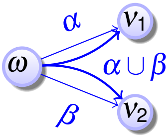
$$P,Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P \rightarrow Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

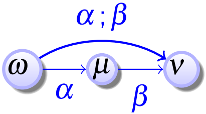


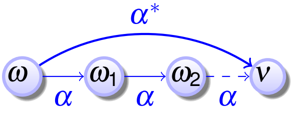
Definition (Hybrid program)

$$\alpha,\beta \ ::= \ x := e \mid ?Q \mid x' = f(x)\,\&\,Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

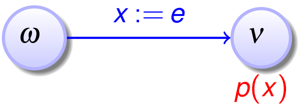






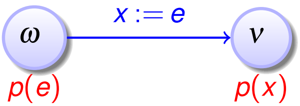


Dynamic Axioms for Dynamical Systems



[:=] $[x := e]p(x) \leftrightarrow$

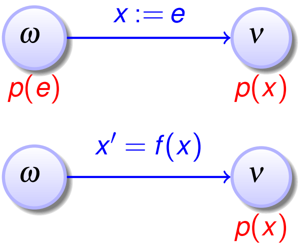
$[:=]$ $[x := e]p(x) \leftrightarrow p(e)$



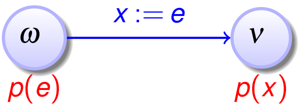
Dynamic Axioms for Dynamical Systems

$[:=]\ [x := e]p(x) \leftrightarrow p(e)$

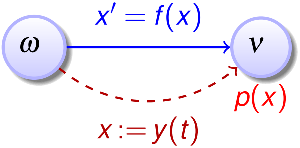
$[']\ [x' = f(x)]p(x) \leftrightarrow$



[:=] $[x := e]p(x) \leftrightarrow p(e)$

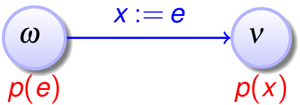


['] $[x' = f(x)]p(x) \leftrightarrow \qquad [x := y(t)]p(x)$

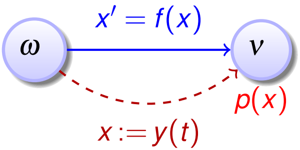


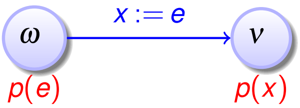
Dynamic Axioms for Dynamical Systems

[:=] $[x := e]p(x) \leftrightarrow p(e)$

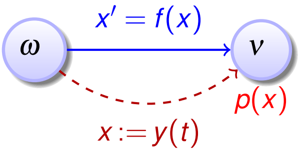


['] $[x' = f(x)]p(x) \leftrightarrow \forall t \geq 0\, [x := y(t)]p(x)$





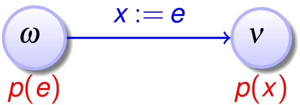
$[:=]\ [x := e]p(x) \leftrightarrow p(e)$



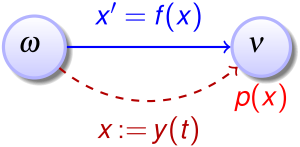
$[']\ [x' = f(x)]p(x) \leftrightarrow \forall t \geq 0\, [x := y(t)]p(x)$

$[']\ [x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t \geq 0\, \big([x := y(t)]p(x)\big)$

$[:=]\ [x := e]p(x) \leftrightarrow p(e)$



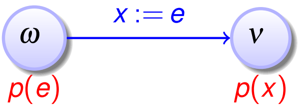
$[']\ [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$



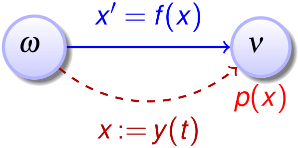
$[']\ [x' = f(x)\,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

Dynamic Axioms for Dynamical Systems

$[:=]\ [x := e]p(x) \leftrightarrow p(e)$

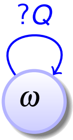


$['\,]\ [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$



$['\,]\ [x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

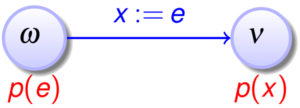
$[?]\ [?Q]P \leftrightarrow$

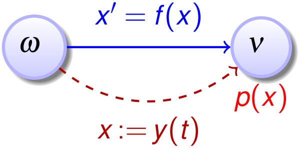


if $\omega \in [\![Q]\!]$

Dynamic Axioms for Dynamical Systems

$[:=]\ [x := e]p(x) \leftrightarrow p(e)$

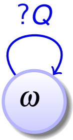




$[']\ [x' = f(x)]p(x) \leftrightarrow \forall t{\geq}0\, [x := y(t)]p(x)$

$[']\ [x' = f(x)\,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big(\forall 0{\leq}s{\leq}t\, q(y(s)) \rightarrow [x := y(t)]p(x)\big)$

$[?]\ [?Q]P \leftrightarrow (Q \rightarrow P)$

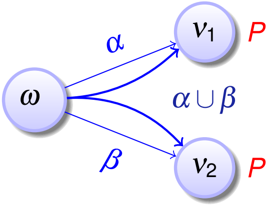


if $\omega \in [\![Q]\!]$

Dynamic Axioms for Dynamical Systems

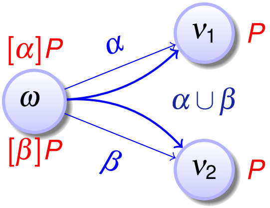
compositional semantics $\Rightarrow$ compositional proofs

[∪] $[\alpha \cup \beta]P \leftrightarrow$

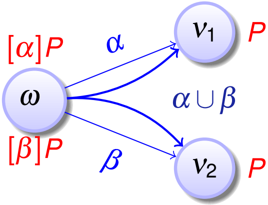


Dynamic Axioms for Dynamical Systems

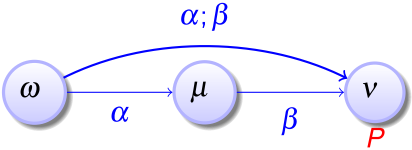
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



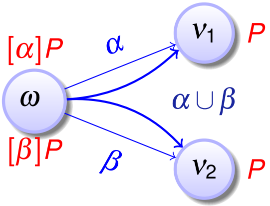
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



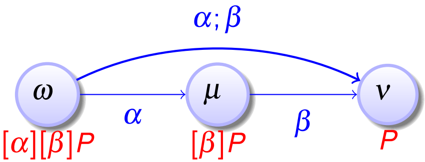
$[;]$ $[\alpha;\beta]P \leftrightarrow$



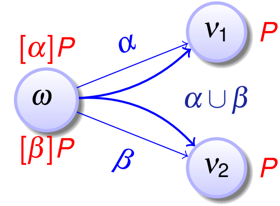
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



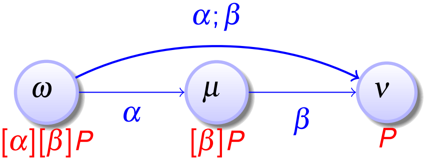
[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



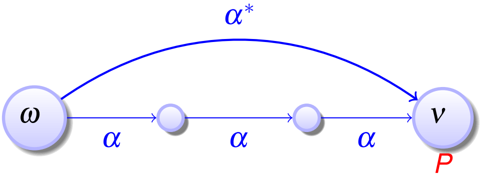
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



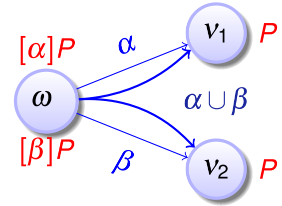
[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



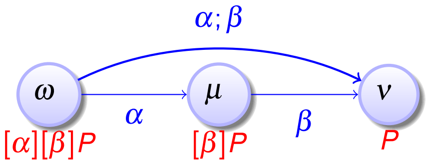
[*] $[\alpha^*]P \leftrightarrow$



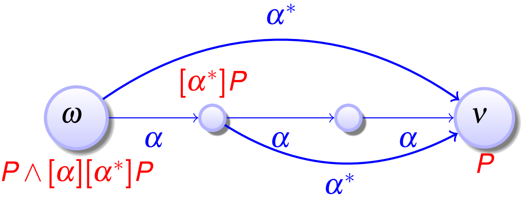
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



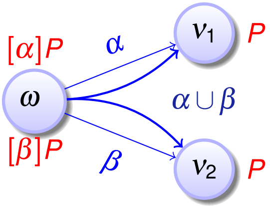
[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



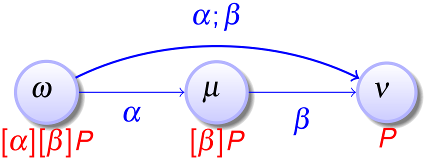
[*] $[\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$



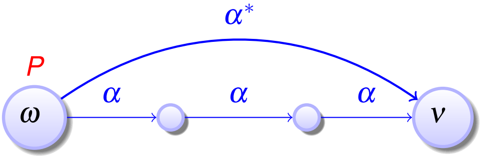
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

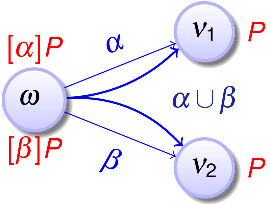


I $[\alpha^*]P \leftrightarrow P \wedge$

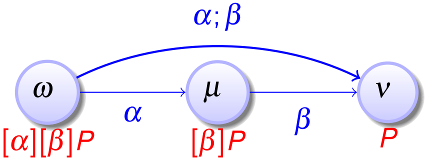


Dynamic Axioms for Dynamical Systems

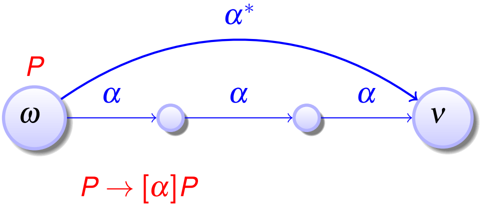
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



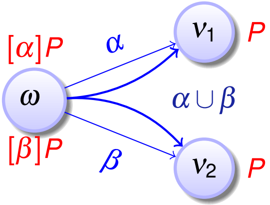
$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



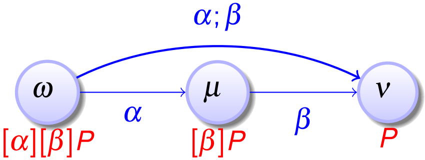
I $[\alpha^*]P \leftrightarrow P \wedge \qquad (P \rightarrow [\alpha]P)$



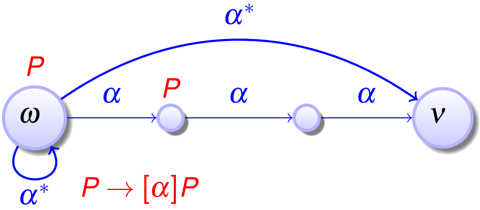
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

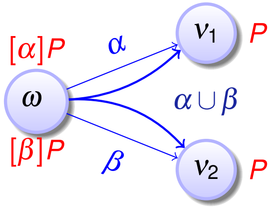


I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

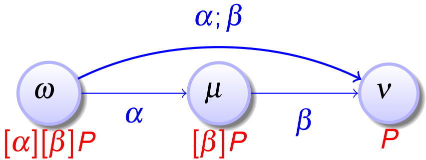


Dynamic Axioms for Dynamical Systems

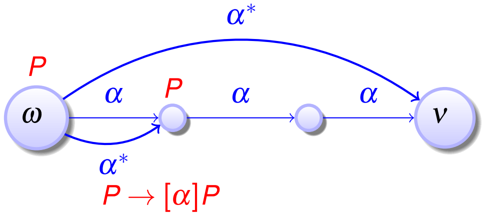
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

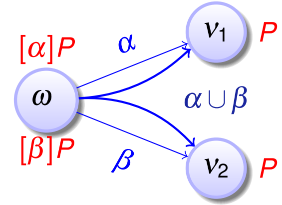


I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \to [\alpha]P)$

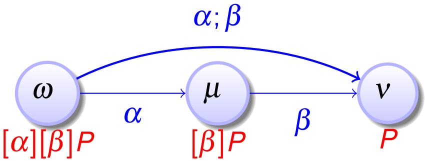


Dynamic Axioms for Dynamical Systems

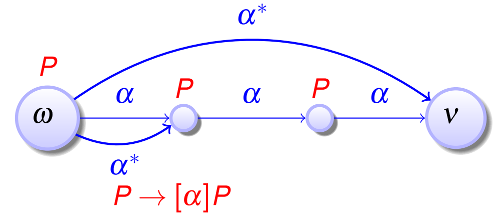
$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



$[;]$ $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

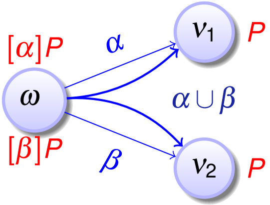


I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

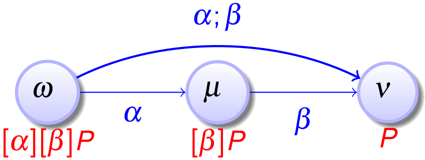


Dynamic Axioms for Dynamical Systems

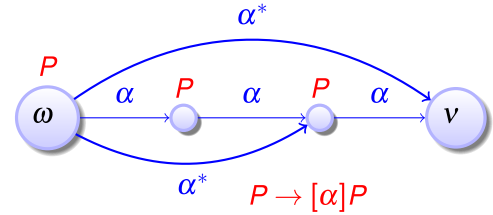
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

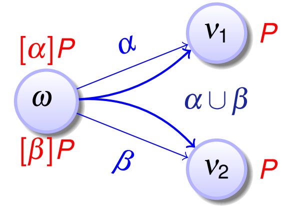


I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$

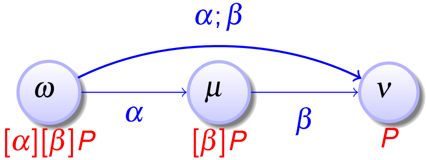


Dynamic Axioms for Dynamical Systems

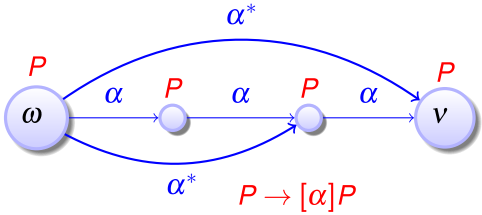
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

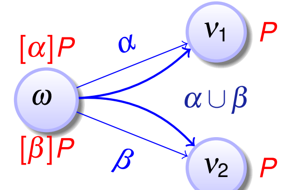


I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

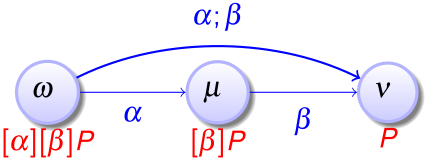


Dynamic Axioms for Dynamical Systems

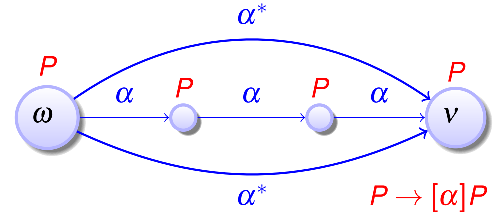
[∪] $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$



[;] $[\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$



I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \to [\alpha]P)$

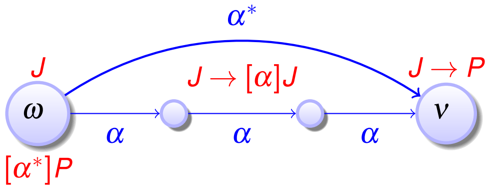


Proof Rule: Loop Invariants

G $\dfrac{P}{[\alpha]P}$ I $[\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$ M[·] $\dfrac{P \rightarrow Q}{[\alpha]P \rightarrow [\alpha]Q}$

Lemma (Loop invariant rule is derived syntactically)

$$\text{loop}\ \frac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$



Sequent notation $\Gamma \vdash \Delta$ means $\left(\bigwedge_{A\in\Gamma} A\right) \rightarrow \left(\bigvee_{B\in\Delta} B\right)$ for sets Γ, Δ

Proof Rule: Loop Invariants

$$\text{G}\ \frac{P}{[\alpha]P} \qquad \text{I}\ [\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \to [\alpha]P) \qquad \text{M}[\cdot]\ \frac{P \to Q}{[\alpha]P \to [\alpha]Q}$$

Lemma (Loop invariant rule is derived syntactically)

$$\text{loop}\ \frac{\Gamma \to J, \Delta \quad J \to [\alpha]J \quad J \to P}{\Gamma \to [\alpha^*]P, \Delta}$$

Proof (Derived rule).

$$\text{cut}\ \dfrac{\Gamma \to J, \Delta \qquad \text{I}\ \dfrac{\text{G}\ \dfrac{J \to [\alpha]J}{J \to J \wedge [\alpha^*](J \to [\alpha]J)}}{J \to [\alpha^*]J} \qquad \text{M}[\cdot]\ \dfrac{J \to P}{[\alpha^*]J \to [\alpha^*]P}}{\Gamma \to [\alpha^*]P, \Delta}$$

□

Proof Rule: Loop Invariants

$$\text{G}\ \frac{P}{[\alpha]P} \qquad \text{I}\ [\alpha^*]P \leftrightarrow P \land [\alpha^*](P \to [\alpha]P) \qquad \text{M}[\cdot]\ \frac{P \to Q}{[\alpha]P \to [\alpha]Q}$$

Lemma (Loop invariant rule is derived syntactically)

$$\text{loop}\ \frac{\Gamma \to J, \Delta \quad J \to [\alpha]J \quad J \to P}{\Gamma \to [\alpha^*]P, \Delta}$$

Proof (Derived rule).

$$\text{cut}\ \dfrac{\Gamma \to J, \Delta \qquad \text{I}\ \dfrac{\text{G}\ \dfrac{J \to [\alpha]J}{J \to J \land [\alpha^*](J \to [\alpha]J)}}{J \to [\alpha^*]J} \qquad \text{M}[\cdot]\ \dfrac{J \to P}{[\alpha^*]J \to [\alpha^*]P}}{\Gamma \to [\alpha^*]P, \Delta}$$

□

Finding invariant J can be a challenge.

Misplaced $[\alpha^*]$ suggests that J needs to carry along info about α^* history.

Differential Dynamic Logic dL: Axiomatization

equations of truth

$[:=]$ $[x := e]P(x) \leftrightarrow P(e)$

$[?]$ $[?Q]P \leftrightarrow (Q \rightarrow P)$

$[']$ $[x' = f(x)]P \leftrightarrow \forall t \geq 0\, [x := y(t)]P$ $\qquad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

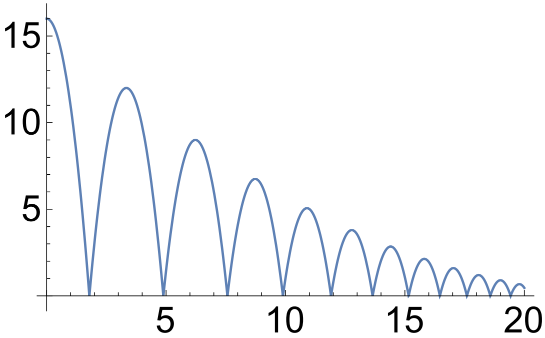
K $[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

C $[\alpha^*]\forall v{>}0\,(P(v) \rightarrow \langle\alpha\rangle P(v-1)) \rightarrow \forall v\,(P(v) \rightarrow \langle\alpha^*\rangle \exists v{\leq}0\, P(v))$

laws of logic of
laws of physics

LICS'12,JAR'17



Example (Bouncing Ball)

$$v{=}0 \land 1{\geq}c{\geq}0 \land H{=}x{\geq}0 \land g{>}0 \rightarrow \big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\};\ \text{if}(x = 0)\, v := -cv\big)^*\big]\, 0{\leq}x{\leq}H$$

$$\frac{}{A \to [\big(\text{grav}; (?x{=}0; v{:=}{-cv} \cup ?x{\neq}0)\big)^*] B(x,v)}$$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$\text{loop}\ \dfrac{A \rightarrow j(x,v) \qquad \dfrac{}{j(x,v) \rightarrow [\text{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0)]j(x,v)} \qquad j(x,v) \rightarrow B(x,v)}{A \rightarrow [(\text{grav}; (?x{=}0; v{:=}{-}cv \cup ?x{\neq}0))^*]B(x,v)}$$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$[:=]\ \frac{\mathrm{j}(x,v),\, x=0 \to \mathrm{j}(x,-cv)}{\mathrm{j}(x,v),\, x=0 \to [v:=-cv]\mathrm{j}(x,v)}$$

$$[?],\to R\ \frac{\mathrm{j}(x,v),\, x=0 \to [v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0][v:=-cv]\mathrm{j}(x,v)}$$

$$[;]\ \frac{\mathrm{j}(x,v) \to [?x=0][v:=-cv]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v)}$$

$$[?]\ \frac{\mathrm{j}(x,v),\, x\neq 0 \to \mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x\neq 0]\mathrm{j}(x,v)}$$

$$\wedge R\ \frac{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x\neq 0]\mathrm{j}(x,v)}$$

$$[\cup]\ \frac{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv]\mathrm{j}(x,v) \wedge [?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}$$

$$\mathrm{MR}\ \frac{\mathrm{j}(x,v) \to [\mathrm{grav}]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}$$

$$[;]\ \frac{\mathrm{j}(x,v) \to [\mathrm{grav}][?x=0;\, v:=-cv \cup ?x\neq 0]\mathrm{j}(x,v)}{\mathrm{j}(x,v) \to [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v)}$$

$$\mathrm{loop}\ \frac{A \to \mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to [\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)]\mathrm{j}(x,v) \qquad \mathrm{j}(x,v) \to B(x,v)}{A \to \left[\left(\mathrm{grav};\, (?x=0;\, v:=-cv \cup ?x\neq 0)\right)^*\right]B(x,v)}$$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\mathrm{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$A \to j(x,v)$

$j(x,v) \to [\text{grav}](j(x,v))$

$j(x,v), x=0 \to j(x,(-cv))$

$j(x,v), x\neq 0 \to j(x,v)$

$j(x,v) \to B(x,v)$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \land x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0 \rightarrow \mathrm{j}(x,v)$$
$$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \geq 0\}](\mathrm{j}(x,v))$$
$$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$$
$$\mathrm{j}(x,v), x \neq 0 \rightarrow \mathrm{j}(x,v)$$
$$\mathrm{j}(x,v) \rightarrow 0 \leq x \land x \leq H$$

$$A \equiv 0 \leq x \land x = H \land v = 0 \land g > 0 \land 1 \geq c \geq 0$$
$$B(x,v) \equiv 0 \leq x \land x \leq H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \to j(x,v)$$
$$j(x,v) \to [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$
$$j(x,v), x=0 \to j(x,(-cv))$$
$$j(x,v), x \ne 0 \to j(x,v)$$
$$j(x,v) \to 0 \le x \land x \le H$$

2. $j(x,v) \equiv 0 \le x \land x \le H$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x\ge 0\}](j(x,v))$$
$$j(x,v), x=0 \rightarrow j(x,(-cv))$$
$$j(x,v), x\ne 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow 0 \le x \wedge x \le H$$

2. $j(x,v) \equiv 0 \le x \wedge x \le H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \le x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \wedge x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$

$j(x,v), x=0 \rightarrow j(x,(-cv))$

$j(x,v), x \neq 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ weak: fails ODE if $v \gg 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \to j(x,v)$$
$$j(x,v) \to [\{x'=v, v'=-g \,\&\, x\ge 0\}](j(x,v))$$
$$j(x,v), x=0 \to j(x,(-cv))$$
$$j(x,v), x\ne 0 \to j(x,v)$$
$$j(x,v) \to 0 \le x \land x \le H$$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow j(x,v)$$

$$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$

$$j(x,v), x=0 \rightarrow j(x,(-cv))$$

$$j(x,v), x \ne 0 \rightarrow j(x,v)$$

$$j(x,v) \rightarrow 0 \le x \land x \le H$$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$

$$B(x,v) \equiv 0 \le x \land x \le H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \geq 0\}](j(x,v))$

$j(x,v), x=0 \rightarrow j(x,(-cv))$

$j(x,v), x \neq 0 \rightarrow j(x,v)$

$j(x,v) \rightarrow 0 \leq x \wedge x \leq H$

1. $j(x,v) \equiv x \geq 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \leq x \wedge x \leq H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \wedge v = 0$ — strong: fails initial condition if $x > 0$

$$A \equiv 0 \leq x \wedge x = H \wedge v = 0 \wedge g > 0 \wedge 1 \geq c \geq 0$$

$$B(x,v) \equiv 0 \leq x \wedge x \leq H$$

$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \geq 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow \mathrm{j}(x,v)$$
$$\mathrm{j}(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](\mathrm{j}(x,v))$$
$$\mathrm{j}(x,v), x=0 \rightarrow \mathrm{j}(x,(-cv))$$
$$\mathrm{j}(x,v), x \ne 0 \rightarrow \mathrm{j}(x,v)$$
$$\mathrm{j}(x,v) \rightarrow 0 \le x \land x \le H$$

1. $\mathrm{j}(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $\mathrm{j}(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $\mathrm{j}(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $\mathrm{j}(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $\mathrm{j}(x,v) \equiv 2gx = 2gH - v^2 \land x \ge 0$

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow [\{x'=v, v'=-g \,\&\, x \ge 0\}](j(x,v))$$
$$j(x,v), x=0 \rightarrow j(x,(-cv))$$
$$j(x,v), x \ne 0 \rightarrow j(x,v)$$
$$j(x,v) \rightarrow 0 \le x \land x \le H$$

1. $j(x,v) \equiv x \ge 0$ — weaker: fails postcondition if $x > H$
2. $j(x,v) \equiv 0 \le x \land x \le H$ — weak: fails ODE if $v \gg 0$
3. $j(x,v) \equiv x = 0 \land v = 0$ — strong: fails initial condition if $x > 0$
4. $j(x,v) \equiv x = 0 \lor x = H \land v = 0$ — no space for intermediate states
5. $j(x,v) \equiv 2gx = 2gH - v^2 \land x \ge 0$ — works: implicitly links v and x

$$A \equiv 0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0$$
$$B(x,v) \equiv 0 \le x \land x \le H$$
$$\text{grav} \equiv \{x' = v, v' = -g \,\&\, x \ge 0\}$$

$0 \le x \land x = H \land v = 0 \land g > 0 \land 1 \ge c \ge 0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

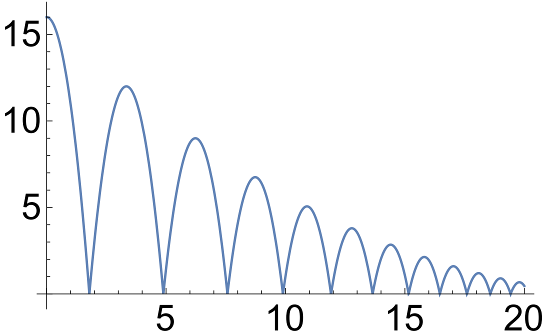
$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow [\{x'{=}v, v'{=}{-}g \,\&\, x{\ge}0\}](2gx{=}2gH{-}v^2 \land x{\ge}0)$

$2gx{=}2gH{-}v^2 \land x{\ge}0, x{=}0 \rightarrow 2gx{=}2gH{-}({-}cv)^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0, x{\neq}0 \rightarrow 2gx{=}2gH{-}v^2 \land x{\ge}0$

$2gx{=}2gH{-}v^2 \land x{\ge}0 \rightarrow 0 \le x \land x \le H$

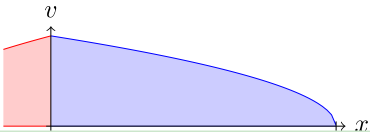
1.
2.
3.
4.
5. $j(x,v) \equiv 2gx{=}2gH{-}v^2 \land x{\ge}0$ works: implicitly links v and x



Example (Bouncing Ball)

$$v=0 \land 1\geq c\geq 0 \land H=x\geq 0 \land g>0 \rightarrow \big[\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v:=-cv\big)^*\big]\, 0\leq x\leq H$$

$Q \equiv 2b(m-x) \geq v^2 + (A+b)(A\varepsilon^2 + 2\varepsilon v)$

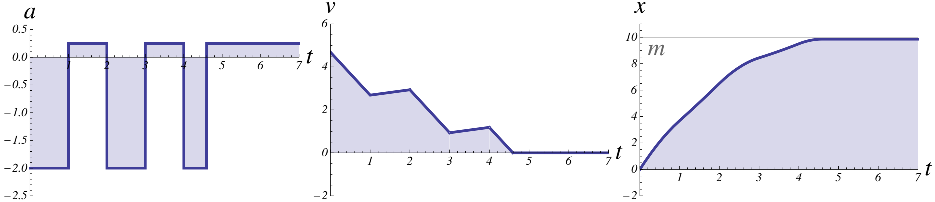


Example (Single car car_ε time-triggered)

$$\Big(\big((?Q; a:=A) \cup a:=-b\big); t:=0; \{x'=v, v'=a, t'=1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}\Big)^*$$

Example (Safely stays before traffic light m)

$$v^2 \leq 2b(m-x) \wedge A \geq 0 \wedge b > 0 \rightarrow [car_\varepsilon]\, x \leq m$$

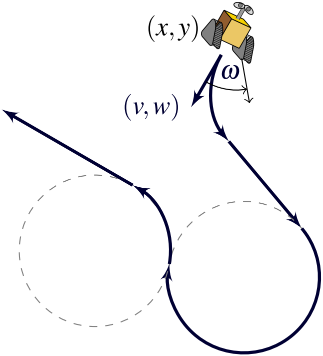


The lion's share of understanding comes from understanding what does change (variants/progress measures) and what doesn't change (invariants).

Invariants are a fundamental force of CS

Variants are another fundamental force of CS

"Making something variable is easy.
Controlling duration of constancy is the trick." – Alan J. Perlis



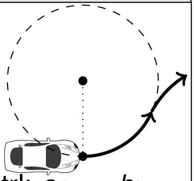
Example (Runaround Robot)

$$(x,y) \neq o \rightarrow \Big[\Big(\big(?Q_{-1};\, \omega:=-1 \cup ?Q_1;\, \omega:=1 \cup ?Q_0;\, \omega:=0\big);\\ \{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\Big)^*\Big]\, (x,y) \neq o$$

Proof search →

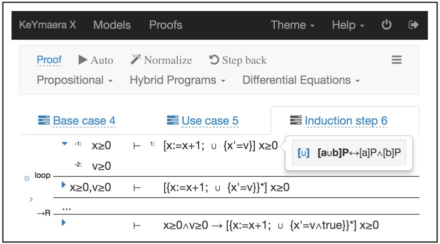
KeYmaera X

Model



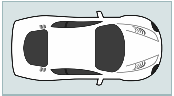
ctrl: $a := -b;$

plant: $x'' = a$



generates proofs

ModelPlex proof →



Model Safety

Compliance Monitor

Trustworthy

Uniform substitution

Sound & complete

Small core: 1700 LOC

Flexible

Proof automation

Interactive UI

Programmable

Customizable

Scala+Java API

Command line

REST API

Theorem (Soundness) replace all occurrences of $p(\cdot)$

$$US\ \frac{\phi}{\sigma(\phi)}$$

provided $FV(\sigma|_{\Sigma(\theta)}) \cap BV(\otimes(\cdot)) = \emptyset$ *for each operation* $\otimes(\theta)$ *in* ϕ

i.e. bound variables $U = \mathrm{BV}(\otimes(\cdot))$ of **no** operator $\otimes$
are free in the substitution on its argument θ (U-admissible)

$$\text{US}\ \frac{[a \cup b]p(\bar{x}) \leftrightarrow [a]p(\bar{x}) \wedge [b]p(\bar{x})}{[x := x+1 \cup x' = 1]x \geq 0 \leftrightarrow [x := x+1]x \geq 0 \wedge [x' = 1]x \geq 0}$$

Uniform Substitution

Theorem (Soundness) replace all occurrences of $p(\cdot)$

$$\text{US}\ \frac{\phi}{\sigma(\phi)}$$

provided $FV(\sigma|_{\Sigma(\theta)}) \cap BV(\otimes(\cdot)) = \emptyset$ *for each operation* $\otimes(\theta)$ *in* ϕ

i.e. bound variables $U = \mathrm{BV}(\otimes(\cdot))$ of **no** operator $\otimes$
are free in the substitution on its argument θ (U-admissible)

$$\frac{[v := f]p(v) \leftrightarrow p(f)}{[v := -x][x' = v]\,x \geq 0 \leftrightarrow [x' = -x]\,x \geq 0}$$

Uniform Substitution

Theorem (Soundness) replace all occurrences of $p(\cdot)$

Modular interface: Prover vs. Logic

$$US \ \frac{\phi}{\sigma(\phi)}$$

provided $FV(\sigma|_{\Sigma(\theta)}) \cap BV(\otimes(\cdot)) = \emptyset$ *for each operation* $\otimes(\theta)$ *in* ϕ

i.e. bound variables $U = \text{BV}(\otimes(\cdot))$ of **no** operator $\otimes$
are free in the substitution on its argument θ (U-admissible)

If you bind a free variable, you go to logic jail!

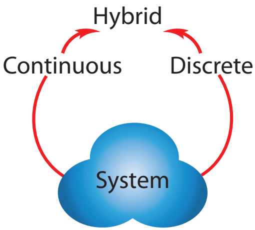
$$\frac{[v := f]p(v) \leftrightarrow p(f)}{[v := -x][x' = v]x \geq 0 \leftrightarrow [x' = -x]x \geq 0}$$

Clash

Complete Proof Theory of Hybrid Systems

Theorem (Sound & Complete) (JAR'08, LICS'12, JAR'17)

dL *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* ***or*** *to discrete dynamics.*



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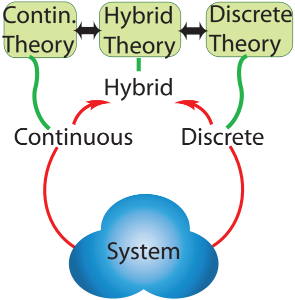


Summary: Proving CPS

equations of truth

$[:=]\ [x := e]P(x) \leftrightarrow P(e)$

$[?]\ [?Q]P \leftrightarrow (Q \rightarrow P)$

$[']\ [x' = f(x)]P \leftrightarrow \forall t{\geq}0\,[x{:=}y(t)]P \qquad (y'(t) = f(y))$

$[\cup]\ [\alpha \cup \beta]P \leftrightarrow [\alpha]P \land [\beta]P$

$[;]\ [\alpha;\beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]\ [\alpha^*]P \leftrightarrow P \land [\alpha][\alpha^*]P$

laws of logic of
laws of physics

$\mathrm{K}\ [\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

$\mathrm{I}\ [\alpha^*]P \leftrightarrow P \land [\alpha^*](P \rightarrow [\alpha]P)$

$\mathrm{C}\ [\alpha^*]\forall v{>}0\,(P(v) \rightarrow \langle\alpha\rangle P(v{-}1)) \rightarrow \forall v\,(P(v) \rightarrow \langle\alpha^*\rangle\exists v{\leq}0\,P(v))$

LICS'12,JAR'17

Outline (Proving ODEs)

$[:=]$ $[x := e]P(x) \leftrightarrow P(e)$

equations of truth

$[?]$ $[?Q]P \leftrightarrow (Q \rightarrow P)$

$[']$ $[x' = f(x)]P \leftrightarrow \forall t \geq 0\, [x := y(t)]P$ $\quad (y'(t) = f(y))$

$[\cup]$ $[\alpha \cup \beta]P \leftrightarrow [\alpha]P \wedge [\beta]P$

$[;]$ $[\alpha; \beta]P \leftrightarrow [\alpha][\beta]P$

$[^*]$ $[\alpha^*]P \leftrightarrow P \wedge [\alpha][\alpha^*]P$

K $[\alpha](P \rightarrow Q) \rightarrow ([\alpha]P \rightarrow [\alpha]Q)$

laws of logic of
laws of physics

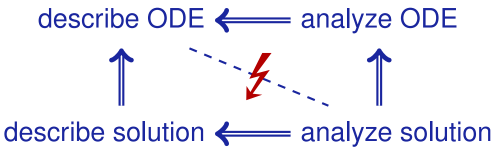
I $[\alpha^*]P \leftrightarrow P \wedge [\alpha^*](P \rightarrow [\alpha]P)$

C $[\alpha^*]\forall v{>}0\, (P(v) \rightarrow \langle\alpha\rangle P(v-1)) \rightarrow \forall v\, (P(v) \rightarrow \langle\alpha^*\rangle \exists v{\leq}0\, P(v))$

LICS'12,JAR'17

Differential Equation Axiomatization

- Classical approach: ① Given ODE ② Solve ODE ③ Analyze solution
- Descriptive power of ODEs: ODE much easier than its solution
- ↯ Analyzing ODEs via their solutions undoes their descriptive power!



Poincaré 1881

1. Logical foundations of differential equation invariants — LICS'18
2. Decide invariance by dL proof

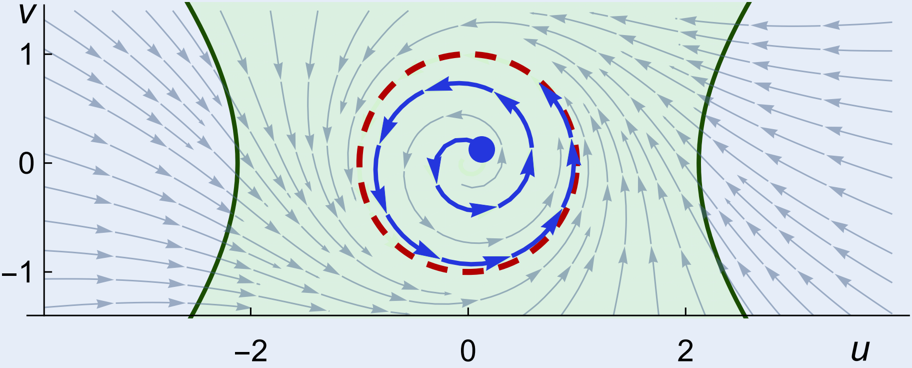
$$x'' = -x \qquad \text{has } x(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots$$

$$x''(t) = e^{t^2} \qquad \text{has no elementary closed-form solution}$$

Concept (Differential Dynamic Logic) (JAR'08,LICS'12)

$$u^2 \leq v^2 + \frac{9}{2} \rightarrow [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2)]\, u^2 \leq v^2 + \frac{9}{2}$$

$$u^2 + v^2 = 1 \rightarrow [u' = -v + \frac{u}{4}(1-u^2-v^2), v' = u + \frac{v}{4}(1-u^2-v^2)]\, u^2 + v^2 = 1$$



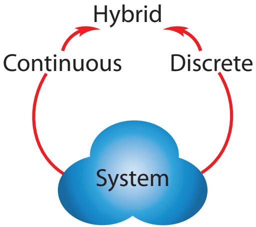
Differential Equation Axiomatization

Theorem (Invariant Completeness) (LICS'18)

dL *calculus is a sound & complete axiomatization of arithmetic invariants of differential equations. They are decidable with a derived axiom.*

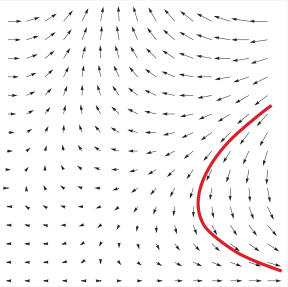
Theorem (Sound & Complete) (JAR'08, LICS'12, JAR'17)

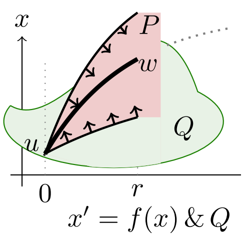
dL *calculus is a sound & complete axiomatization of hybrid systems relative to either differential equations* ***or*** *to discrete dynamics.*



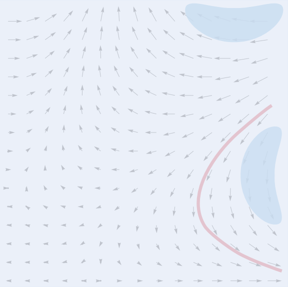
Differential Invariants for Differential Equations

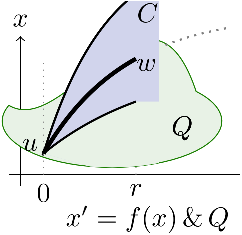
Differential Invariant



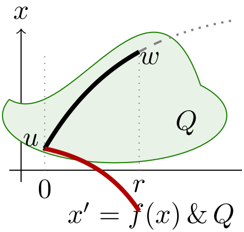


Differential Cut



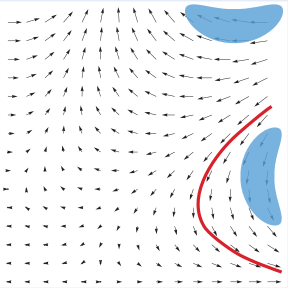


Differential Ghost

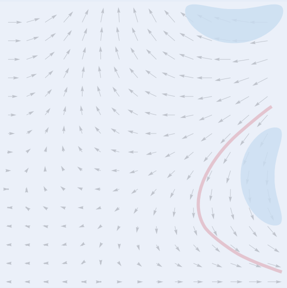


Differential Invariants for Differential Equations

Differential Invariant



Differential Cut



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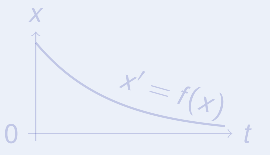
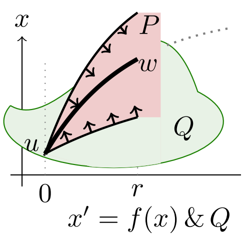
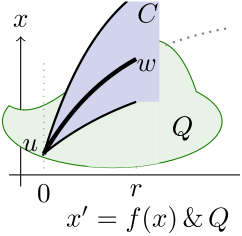
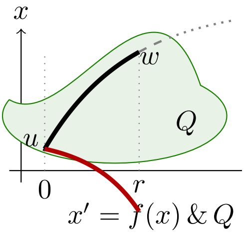


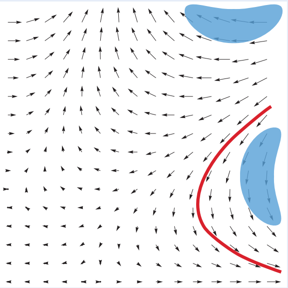


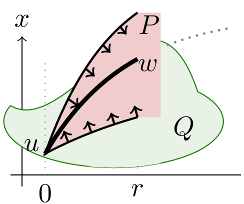




Differential Invariants for Differential Equations

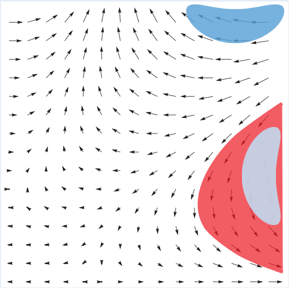
Differential Invariant

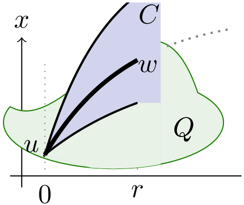




$x' = f(x) \,\&\, Q$

Differential Cut





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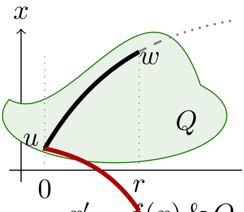
Differential Ghost



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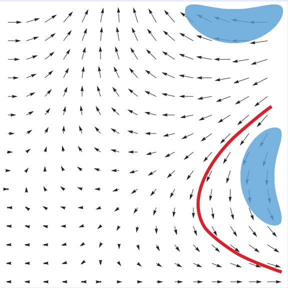
Differential Invariants for Differential Equations

Differential Invariant



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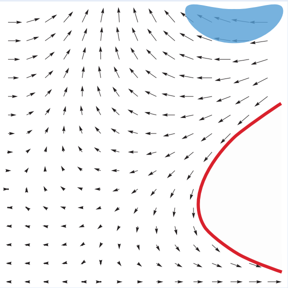

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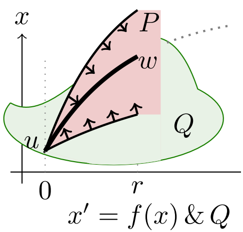
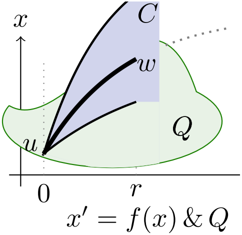
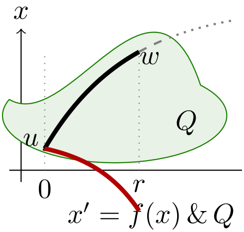
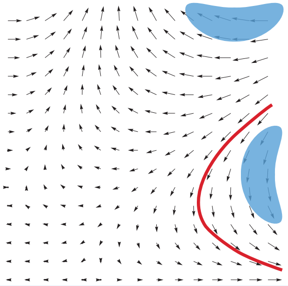




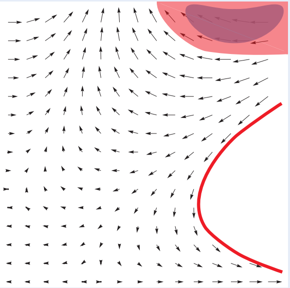


Differential Invariants for Differential Equations

Differential Invariant



Differential Cut



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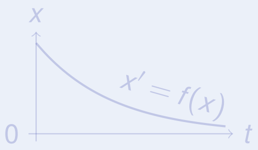
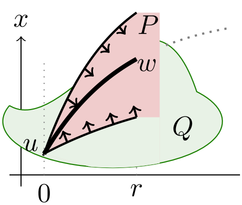
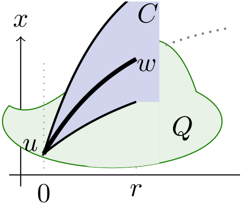
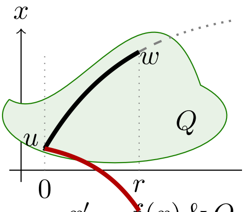




$x' = f(x) \,\&\, Q$



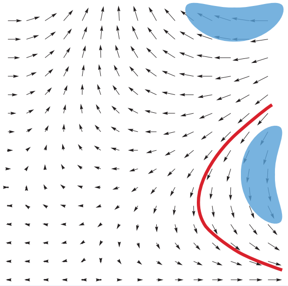
$x' = f(x) \,\&\, Q$



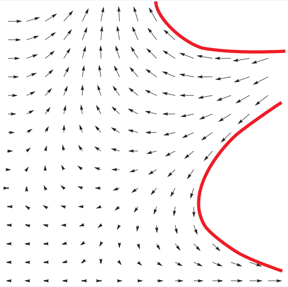
$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

Differential Invariant



Differential Cut



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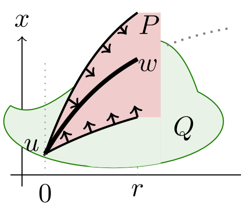

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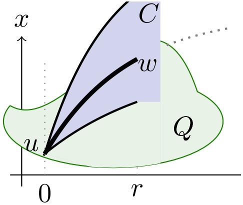

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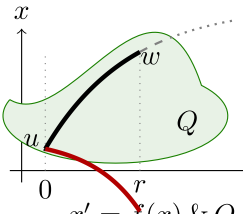
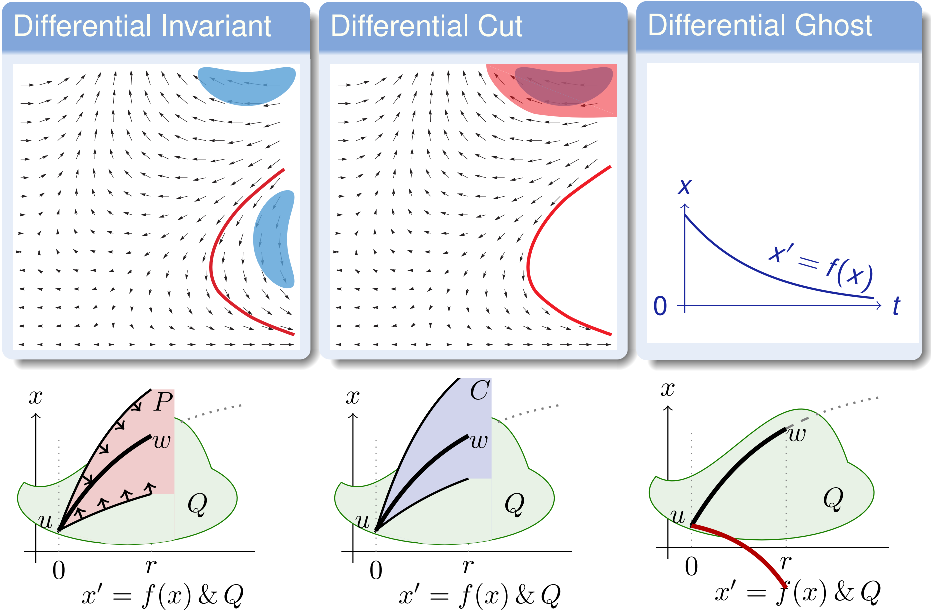


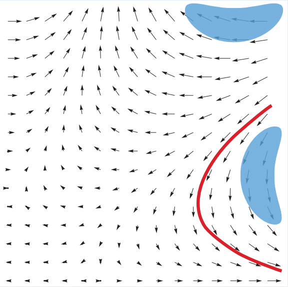
$x' = f(x) \,\&\, Q$

Differential Invariants for Differential Equations

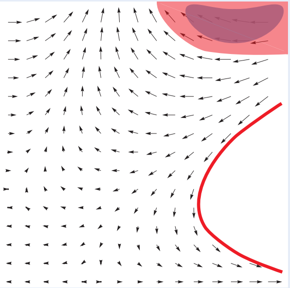


Differential Invariants for Differential Equations

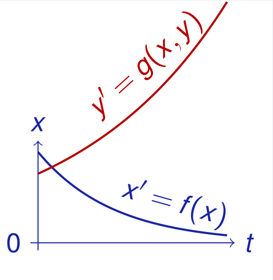
Differential Invariant

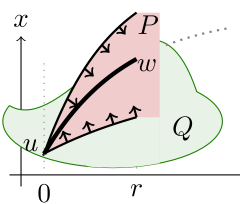


Differential Cut

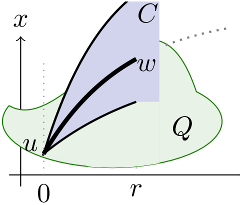


Differential Ghost

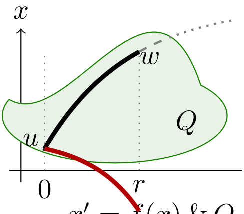




$x' = f(x) \,\&\, Q$



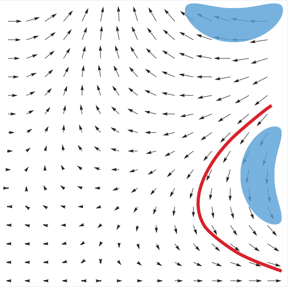
$x' = f(x) \,\&\, Q$

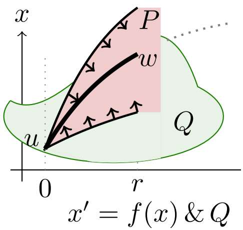


$x' = f(x) \,\&\, Q$

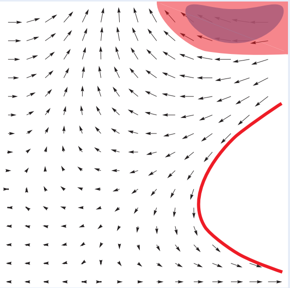
Differential Invariants for Differential Equations

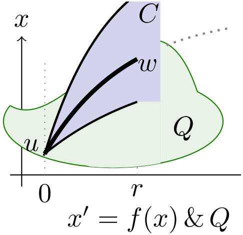
Differential Invariant



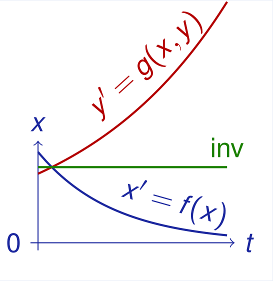


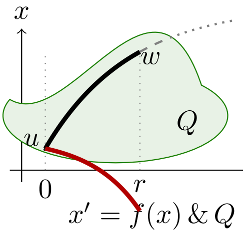
Differential Cut





Differential Ghost

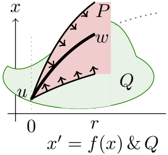




Differential Invariants for Differential Equations

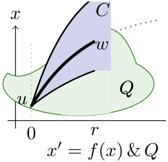
Differential Invariant

$$\frac{Q \rightarrow [x' := f(x)](P)'}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



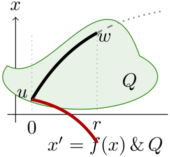
Differential Cut

$$\frac{P \rightarrow [x' = f(x) \,\&\, Q]C \quad P \rightarrow [x' = f(x) \,\&\, Q \wedge C]P}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \rightarrow [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



deductive power added DI $\prec$ DI+DC $\prec$ DI+DC+DG

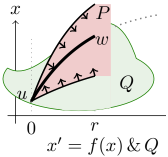
$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Springer'10, LMCS'12, LICS'12,JAR'17,LICS'18,JACM'20

Differential Invariants for Differential Equations

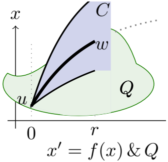
Differential Invariant

$$\frac{Q \rightarrow [x' := f(x)](P)'}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



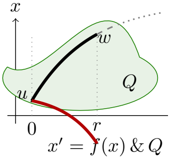
Differential Cut

$$\frac{P \rightarrow [x' = f(x) \,\&\, Q]C \quad P \rightarrow [x' = f(x) \,\&\, Q \land C]P}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



Differential Ghost

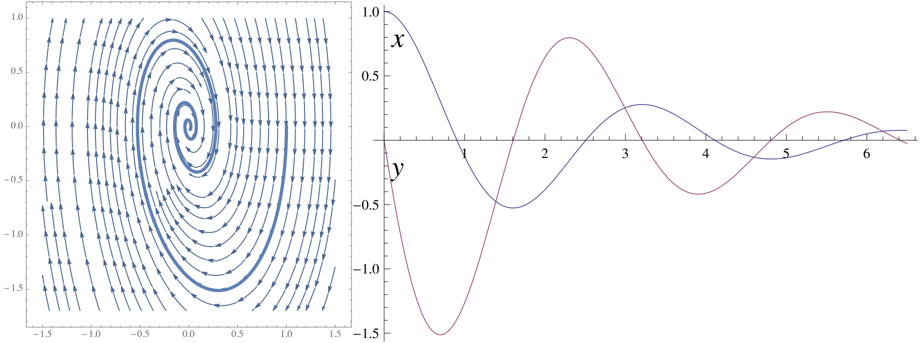
$$\frac{P \leftrightarrow \exists y\, G \quad G \rightarrow [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$



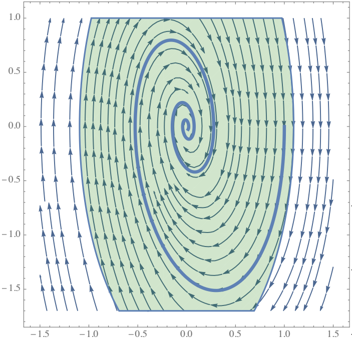
if $g(x,y) = a(x)y + b(x)$, so has long solution!

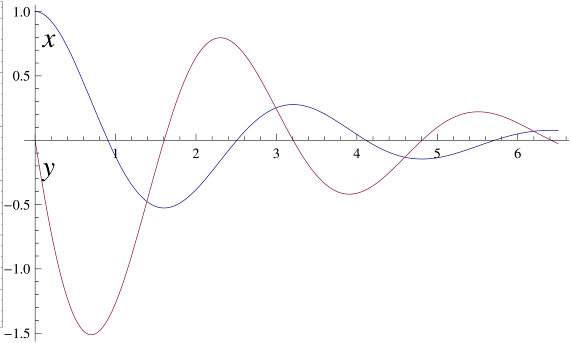
Springer'10, LMCS'12, LICS'12,JAR'17,LICS'18,JACM'20

$$\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y \,\&\, \omega\geq 0 \wedge d\geq 0]\,\omega^2x^2+y^2\leq c^2$$



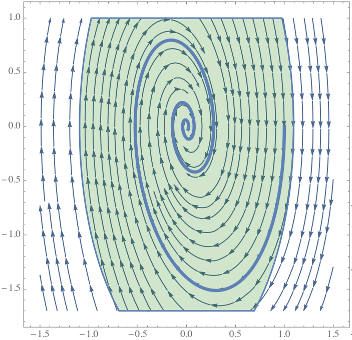
$$\omega^2x^2+y^2\le c^2 \rightarrow [x' = y, y' = -\omega^2x - 2d\omega y \,\&\, \omega\ge 0 \land d\ge 0]\, \omega^2x^2+y^2\le c^2$$

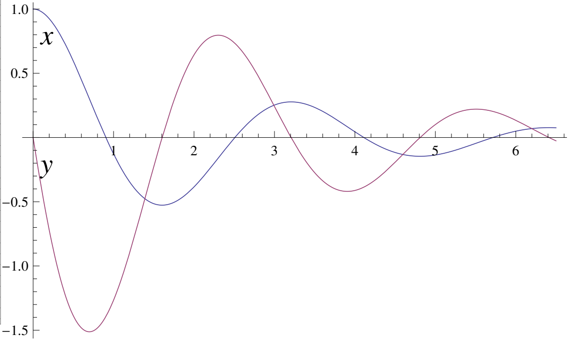




damped oscillator

$$\frac{\omega\geq 0 \land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2 y y' \leq 0}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}$$



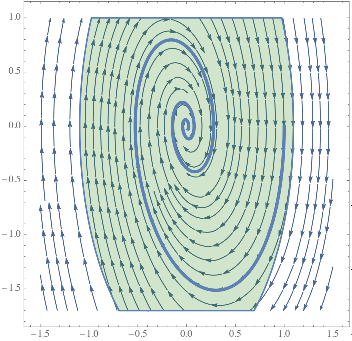


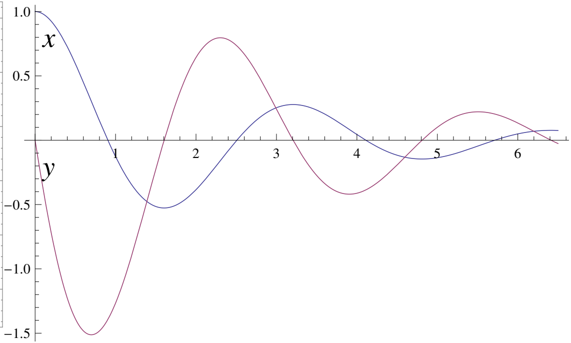
damped oscillator

$$\omega\geq 0 \land d\geq 0 \rightarrow 2\omega^2 x y + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega\geq 0 \land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$





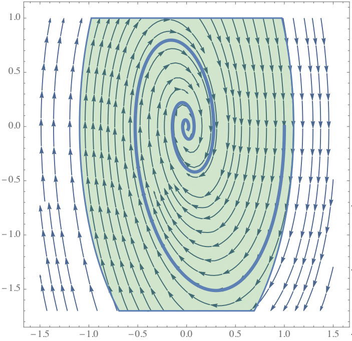
damped oscillator

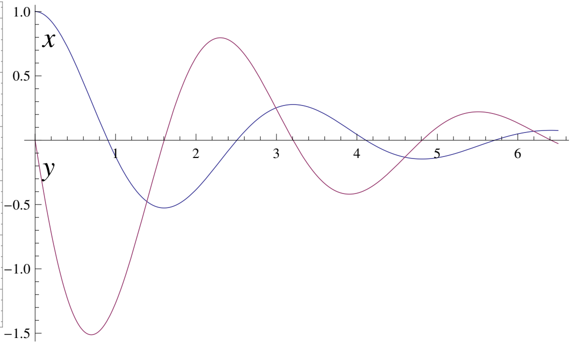
$$*$$

$$\omega\geq 0 \land d\geq 0 \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0$$

$$\omega\geq 0 \land d\geq 0 \rightarrow [x' := y][y' := -\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0$$

$$\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2$$





damped oscillator

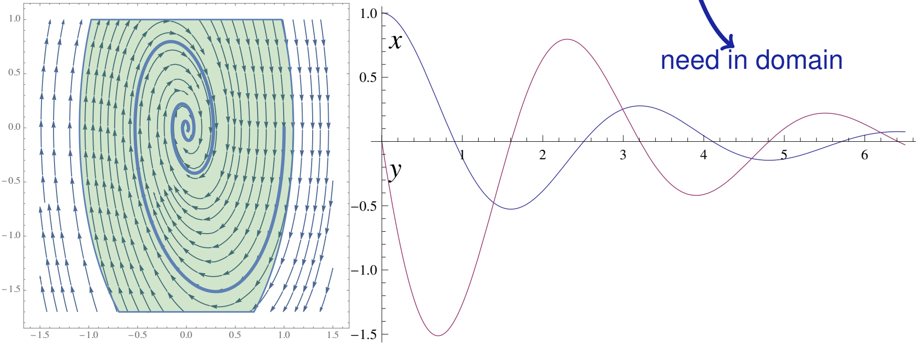
$$\ast$$

$$\omega\geq0\land d\geq0 \to 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0$$

$$\omega\geq0\land d\geq0 \to [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0$$

$$\omega^2x^2+y^2\leq c^2 \to [x'=y,y'=-\omega^2x-2d\omega y\;\&\;\omega\geq0\land d\geq0]\,\omega^2x^2+y^2\leq c^2$$

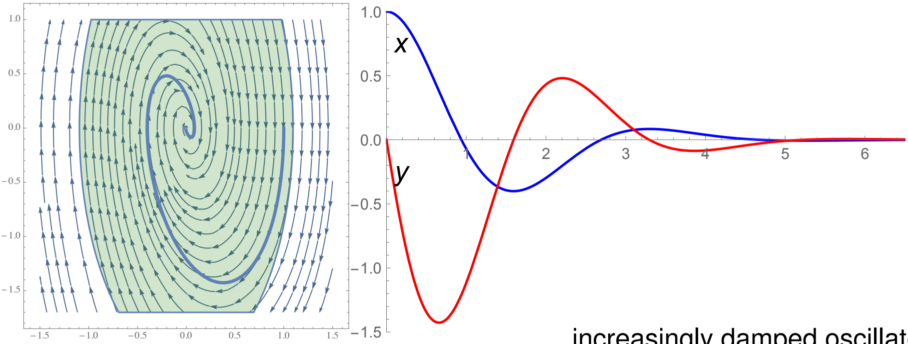
need in domain



damped oscillator

$$\frac{}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \,\&\, \omega \geq 0]\, \omega^2 x^2 + y^2 \leq c^2}$$

$$\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, \omega^2x^2+y^2\le c^2$$



increasingly damped oscillator

$$\frac{\omega^2 x^2 + y^2 \le c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \,\&\, \omega \ge 0 \wedge d \ge 0]\, \omega^2 x^2 + y^2 \le c^2}{\omega^2 x^2 + y^2 \le c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \,\&\, \omega \ge 0]\, \omega^2 x^2 + y^2 \le c^2}$$

increasingly damped oscillator

$$\dfrac{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0\land d\ge 0]\,\omega^2x^2+y^2\le c^2}{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0]\,\omega^2x^2+y^2\le c^2}$$

ask

$$\dfrac{}{d\ge 0 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0]\,d\ge 0}$$

increasingly damped oscillator

$$\dfrac{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0\land d\ge 0]\ \omega^2x^2+y^2\le c^2}{\omega^2x^2+y^2\le c^2 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0]\ \omega^2x^2+y^2\le c^2}$$

$$\dfrac{\overline{\omega\ge 0 \rightarrow [d':=7]\ d'\ge 0}}{d\ge 0 \rightarrow [x'=y, y'=-\omega^2x-2d\omega y, d'=7\ \&\ \omega\ge 0]\ d\ge 0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{}{\omega^2x^2+y^2\le c^2 \to [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0 \land d\ge 0]\, \omega^2x^2+y^2\le c^2}}{\omega^2x^2+y^2\le c^2 \to [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, \omega^2x^2+y^2\le c^2}$$

$$\dfrac{\dfrac{\dfrac{}{\omega\ge 0 \to 7\ge 0}}{\omega\ge 0 \to [d':=7]\, d'\ge 0}}{d\ge 0 \to [x'=y, y'=-\omega^2x-2d\omega y, d'=7 \,\&\, \omega\ge 0]\, d\ge 0}$$

increasingly damped oscillator

$$\omega^2x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0\land d\geq 0]\,\omega^2x^2+y^2\leq c^2$$

$$\overline{\omega^2x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

DC

$$*$$

$$\overline{\omega\geq 0 \rightarrow 7\geq 0}$$

$$\overline{\omega\geq 0 \rightarrow [d':=7]\,d'\geq 0}$$

$$\overline{d\geq 0 \rightarrow [x' = y, y' = -\omega^2x - 2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{}{\omega\geq 0 \land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2y y' \leq 0}}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0 \land d\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}}{\omega^2 x^2 + y^2 \leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2 x^2 + y^2 \leq c^2}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \rightarrow 7\geq 0}}{\omega\geq 0 \rightarrow [d':=7]\, d'\geq 0}}{d\geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\omega\geq 0\land d\geq 0 \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0\land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0\land d\geq 0]\, \omega^2x^2+y^2\leq c^2}}{\omega^2x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, \omega^2x^2+y^2\leq c^2}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \rightarrow 7\geq 0}}{\omega\geq 0 \rightarrow [d':=7]\, d'\geq 0}}{d\geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7 \,\&\, \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

$$\dfrac{\dfrac{\dfrac{\dfrac{*}{\omega\geq 0\land d\geq 0 \rightarrow 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0}}{\omega\geq 0\land d\geq 0 \rightarrow [x':=y][y':=-\omega^2 x - 2d\omega y]\, 2\omega^2 x x' + 2yy' \leq 0}}{\omega^2 x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\ \&\ \omega\geq 0\land d\geq 0]\, \omega^2 x^2+y^2\leq c^2}}{\omega^2 x^2+y^2\leq c^2 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\ \&\ \omega\geq 0]\, \omega^2 x^2+y^2\leq c^2}$$

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \rightarrow 7\geq 0}}{\omega\geq 0 \rightarrow [d':=7]\, d'\geq 0}}{d\geq 0 \rightarrow [x' = y, y' = -\omega^2 x - 2d\omega y, d'=7\ \&\ \omega\geq 0]\, d\geq 0}$$

increasingly damped oscillator

Differential Cuts for Differential Equations

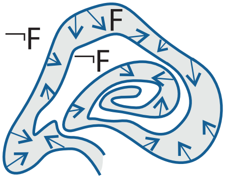
$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0\land d\geq 0 \rightarrow 2\omega^2xy+2y(-\omega^2x-2d\omega y)\leq 0}}{\omega\geq 0\land d\geq 0 \rightarrow [x':=y][y':=-\omega^2x-2d\omega y]\,2\omega^2xx'+2yy'\leq 0}}{\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0\land d\geq 0]\,\omega^2x^2+y^2\leq c^2}$$

$$\omega^2x^2+y^2\leq c^2 \rightarrow [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,\omega^2x^2+y^2\leq c^2$$

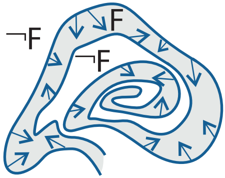
init

$$\dfrac{\dfrac{\dfrac{*}{\omega\geq 0 \rightarrow 7\geq 0}}{\omega\geq 0 \rightarrow [d':=7]\,d'\geq 0}}{d\geq 0 \rightarrow [x'=y,y'=-\omega^2x-2d\omega y, d'=7\,\&\,\omega\geq 0]\,d\geq 0}$$

Could repeatedly diffcut in formulas to help the proof

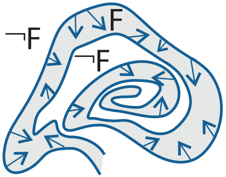


$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

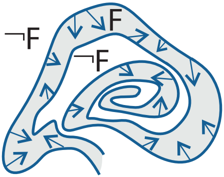


$$\frac{\color{red}{F} \wedge Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

Assuming Differential Invariance



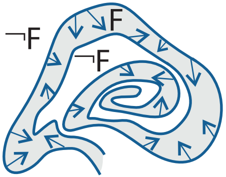
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$



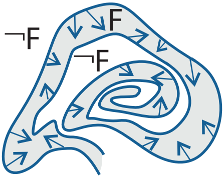
$$\frac{\color{red}{F} \land Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

Example (Inductive hypothesis)

$$\frac{}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]\, v^2 - 2v + 1 = 0}$$



$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

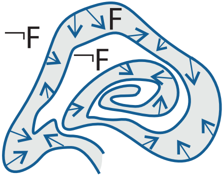


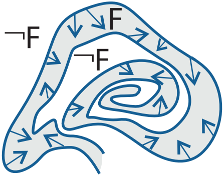
$$\frac{F \land Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

Example (Inductive hypothesis)

$$\frac{\frac{}{v^2 - 2v + 1 = 0 \rightarrow [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance





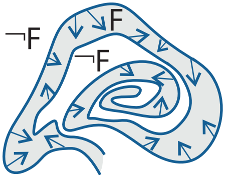
$$\frac{Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

$$\frac{F \wedge Q \rightarrow [x' := f(x)](F)'}{F \rightarrow [x' = f(x) \,\&\, Q]F}$$

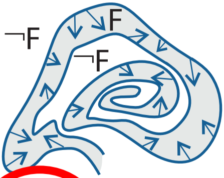
Example (Inductive hypothesis)

$$\frac{\frac{\frac{}{v^2 - 2v + 1 = 0 \rightarrow 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \rightarrow [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

Assuming Differential Invariance



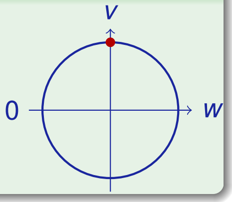
$$\frac{Q \to [x' := f(x)](F)'}{F \to [x' = f(x) \,\&\, Q]F}$$



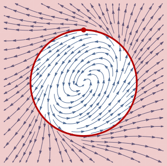
$$\frac{F \land Q \to [x' := f(x)](F)'}{F \to [x' = f(x) \,\&\, Q]F}$$

Example (Inductive hypothesis is unsound!)

$$\dfrac{\dfrac{\dfrac{\text{(unsound)}}{v^2 - 2v + 1 = 0 \to 2vw - 2w = 0}}{v^2 - 2v + 1 = 0 \to [v' := w][w' := -v]2vv' - 2v' = 0}}{v^2 - 2v + 1 = 0 \to [v' = w, w' = -v]v^2 - 2v + 1 = 0}$$

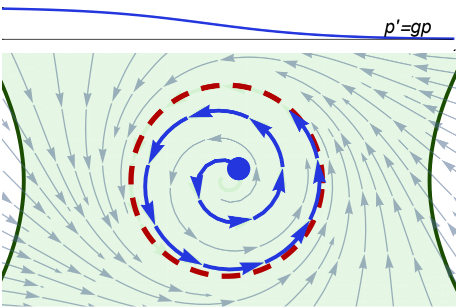


Induction for ODEs is subtle!



Darboux inequalities are DG

$$\frac{Q \rightarrow p^{\bullet} \geq gp}{p \succcurlyeq 0 \rightarrow [x' = f(x) \,\&\, Q] p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$

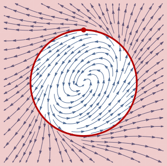


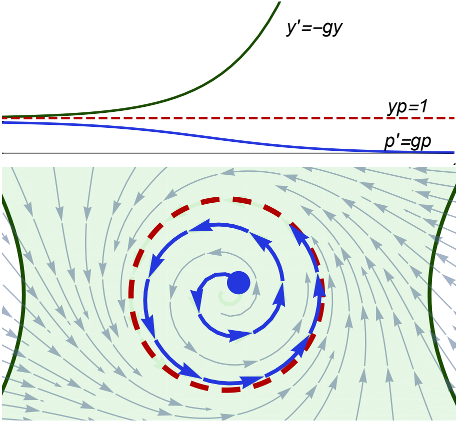
$$\frac{(1-u^2-v^2)^{\bullet} \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \rightarrow \left[\begin{array}{l} u' = -v + \frac{u}{4}(1-u^2-v^2) \\ v' = u + \frac{v}{4}(1-u^2-v^2) \end{array}\right] \underbrace{1-u^2-v^2 > 0}}$$

Definable $p^{\bullet}$ for Lie-derivative w.r.t. ODE

Darboux inequalities are DG

$$\frac{Q \rightarrow p^{\bullet} \geq gp}{p \succcurlyeq 0 \rightarrow [x' = f(x) \,\&\, Q]\, p \succcurlyeq 0} \qquad (g \in \mathbb{R}[x])$$



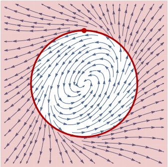


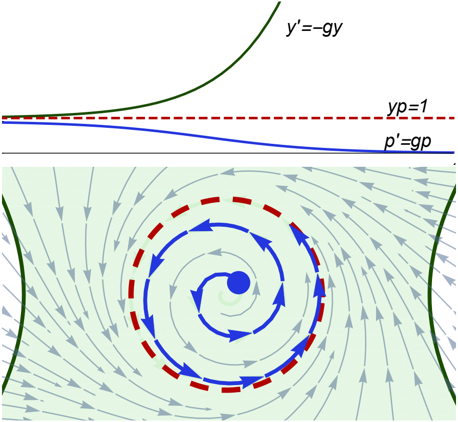
$$\frac{(1-u^2-v^2)^{\bullet} \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \rightarrow \left[\begin{aligned} u' &= -v + \tfrac{u}{4}(1-u^2-v^2) \\ v' &= u + \tfrac{v}{4}(1-u^2-v^2) \\ y' &= \tfrac{1}{2}(u^2+v^2)y \end{aligned}\right] \underbrace{1-u^2-v^2 > 0}_{y(1-u^2-v^2)=1}}$$

Example Proof Derived by Differential Ghosts

Darboux inequalities are DG

$$\frac{Q \to p' \geq gp}{p \succcurlyeq 0 \to [x' = f(x) \,\&\, Q] p \succcurlyeq 0} \quad (g \in \mathbb{R}[x])$$





$$\frac{(1-u^2-v^2)' \geq -\frac{1}{2}(u^2+v^2)(1-u^2-v^2)}{\ldots \to \left[\begin{array}{l} u' = -v + \frac{u}{4}(1-u^2-v^2) \\ v' = u + \frac{v}{4}(1-u^2-v^2) \\ y' = \frac{1}{2}(u^2+v^2)y \\ z' = -\frac{1}{4}(u^2+v^2)z \end{array}\right] \underbrace{1-u^2-v^2 > 0}_{y(1-u^2-v^2)=1}}$$

Example Proof Derived by Differential Ghosts

$$*$$

$\mathbb{R}$ $$Q \to (-gy)z^2 + y(2z(\tfrac{g}{2}z)) = 0$$

dI $$yz^2 = 1 \to [x' = f(x), y' = -gy, z' = \tfrac{g}{2}z \,\&\, Q]\, yz^2 = 1$$

M[·],∃R $$y > 0 \to \exists z\, [x' = f(x), y' = -gy, z' = \tfrac{g}{2}z \,\&\, Q]\, y > 0$$

dG $$y > 0 \to [x' = f(x), y' = -gy \,\&\, Q]\, y > 0$$

$$*$$

$$Q \to p^{\bullet} \geq gp \qquad \mathbb{R}\ \ p^{\bullet} \geq gp, y > 0 \to p^{\bullet} y - gyp \geq 0$$

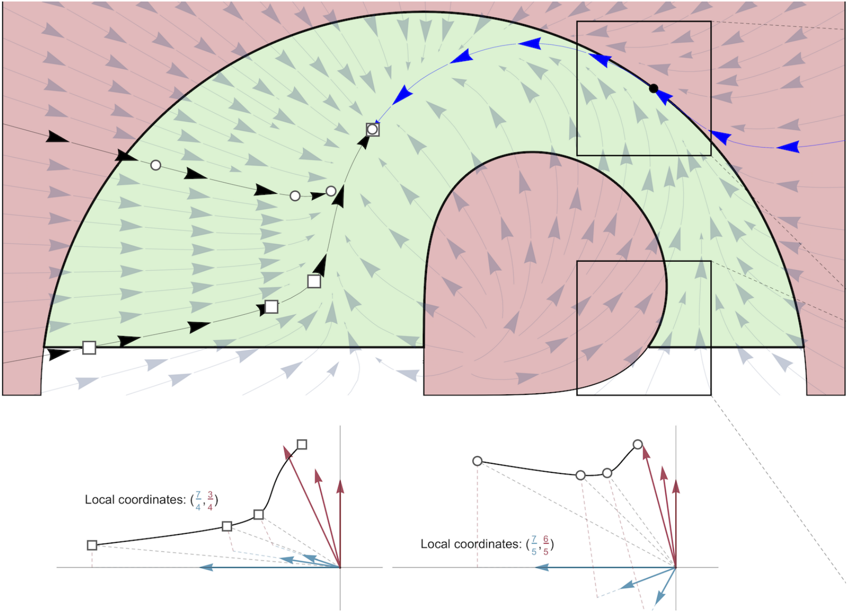
cut $$Q, y > 0 \to p^{\bullet} y - gyp \geq 0$$

dI $$p \succcurlyeq 0, y > 0 \to [x' = f(x), y' = -gy \,\&\, Q \land y > 0]\, py \succcurlyeq 0 \quad \triangleright$$

dC $$p \succcurlyeq 0, y > 0 \to [x' = f(x), y' = -gy \,\&\, Q](y > 0 \land py \succcurlyeq 0)$$

M[·],∃R $$p \succcurlyeq 0 \to \exists y\, [x' = f(x), y' = -gy \,\&\, Q]\, p \succcurlyeq 0$$

dG $$p \succcurlyeq 0 \to [x' = f(x) \,\&\, Q]\, p \succcurlyeq 0$$



LICS'18,JACM'20

Differential Equation Axiomatization

Theorem (Algebraic Completeness) (LICS'18,JACM'20)

dL *calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable by DI,DC,DG in* dL*.*

Theorem (Semialgebraic Completeness) (LICS'18,JACM'20)

dL *calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable in* dL*.*

Differential Equation Axiomatization

Theorem (Algebraic Completeness) (LICS'18,JACM'20)

dL *calculus is a sound & complete axiomatization of algebraic invariants of polynomial differential equations. They are decidable*

$$\text{DRI} \ [x' = f(x) \,\&\, Q]e = 0 \leftrightarrow \big(Q \rightarrow e'^{*} = 0\big) \qquad (Q \text{ open})$$

Theorem (Semialgebraic Completeness) (LICS'18,JACM'20)

dL *calculus with RI is a sound & complete axiomatization of semialgebraic invariants of differential equations. They are decidable*

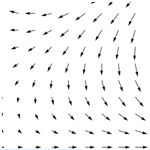
$$\text{SAI} \ \forall x \, (P \rightarrow [x' = f(x)]P) \leftrightarrow \forall x \, \big(P \rightarrow P'^{*}\big) \land \forall x \, \big(\neg P \rightarrow (\neg P)'^{*-}\big)$$

Definable e'^{*} is short for *all/significant* Lie derivative w.r.t. ODE
Definable e'^{*-} is w.r.t. backwards ODE $x' = -f(x)$. Also for P.

$$e'^{*} = 0 \equiv e{=}0 \land (e')'^{*}{=}0 \qquad (P \land Q)'^{*} \equiv P'^{*} \land Q'^{*}$$

$$e'^{*} \geq 0 \equiv e{\geq}0 \land (e{=}0 {\rightarrow} (e')'^{*}{\geq}0) \qquad (P \lor Q)'^{*} \equiv P'^{*} \lor Q'^{*}$$

Differentials



$\to \mathbb{R}$

Syntax $e ::= x \mid x' \mid c \mid e + k \mid e \cdot k \mid {\color{red}(e)'}$

Semantics $$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Axioms

$$(e+k)' = (e)' + (k)'$$
$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$
$$(c())' = 0 \quad \text{for constants/numbers } c()$$
$$(x)' = x' \quad \text{for variables } x \in \mathscr{V}$$

ODE

$$[\![x' = f(x) \,\&\, Q]\!] = \{(\varphi(0)|_{\{x'\}^\complement}, \varphi(r)) : \varphi \models x' = f(x) \land Q$$
$$\text{for some } \varphi : [0,r] \to \mathscr{S}, \text{ some } r \in \mathbb{R}\}$$
$$\varphi(z)(x') = \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z) \qquad \dots$$

JAR'17

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ *for duration* $r>0$, *then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

Syntactic $\rightarrow$ $$\varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z)$$ $\leftarrow$ Analytic

Lemma (Differential assignment) (Effect on Differentials)

DE $[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$

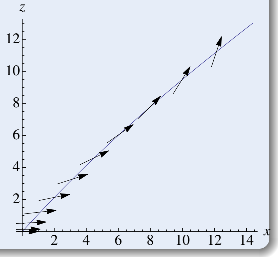
Lemma (Derivations) (Equations of Differentials)

$+'$	$(e+k)' = (e)' + (k)'$
$\cdot'$	$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
c'	$(c())' = 0$
x'	$(x)' = x'$

Example: Longitudinal Dynamics of an Airplane

Study (6th Order Longitudinal Flight Equations)

$u' = \frac{X}{m} - g\sin(\theta) - qw$	axial velocity
$w' = \frac{Z}{m} + g\cos(\theta) + qu$	vertical velocity
$x' = \cos(\theta)u + \sin(\theta)w$	range
$z' = -\sin(\theta)u + \cos(\theta)w$	altitude
$\theta' = q$	pitch angle
$q' = \frac{M}{I_{yy}}$	pitch rate



X : thrust along u Z : thrust along w M : thrust moment for w

g : gravity m : mass I_{yy} : inertia second diagonal

TACAS'14

Example: Longitudinal Dynamics of an Airplane

Study (6th Order Longitudinal Flight Equations)

$$u' = \frac{X}{m} - g\sin(\theta) - qw$$ axial velocity

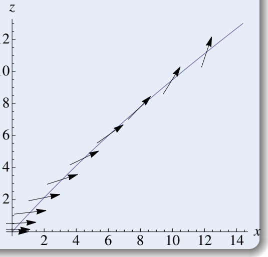
$$w' = \frac{Z}{m} + g\cos(\theta) + qu$$ vertical velocity

$$x' = \cos(\theta)u + \sin(\theta)w$$ range

$$z' = -\sin(\theta)u + \cos(\theta)w$$ altitude

$$\theta' = q$$ pitch angle

$$q' = \frac{M}{I_{yy}}$$ pitch rate

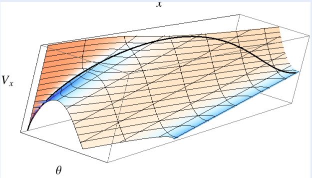


Result (DRI Automatically Generates Invariant Functions)

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$



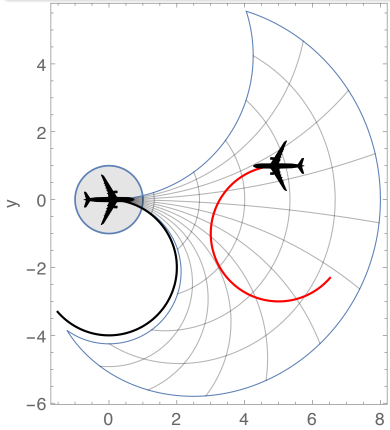
TACAS'14

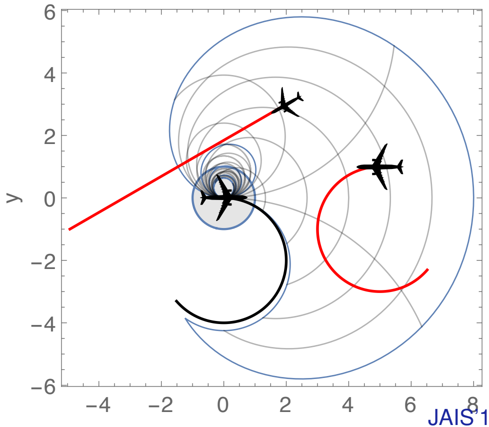
Example: Dubins Dynamics of 2 Airplanes

Result (DRI Automatically Generates Invariants)

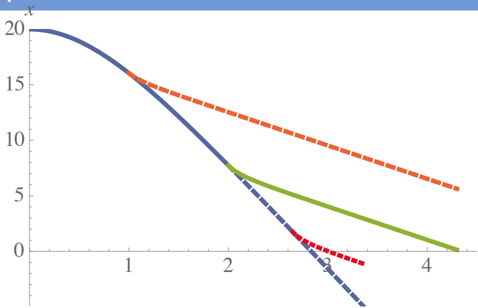
$$\omega_1 = 0 \land \omega_2 = 0 \to v_2 \sin\vartheta x = (v_2 \cos\vartheta - v_1)y > p(v_1 + v_2)$$

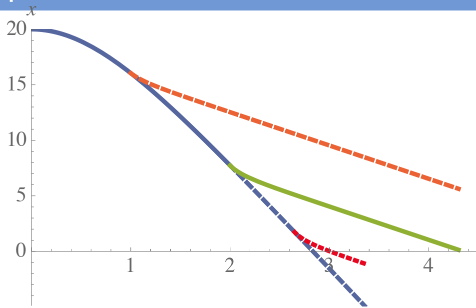
$$\omega_1 \neq 0 \lor \omega_2 \neq 0 \to -\omega_1\omega_2(x^2+y^2) + 2v_2\omega_1 \sin\vartheta x + 2(v_1\omega_2 - v_2\omega_1\cos\vartheta)y + 2v_1v_2\cos\vartheta > 2v_1v_2 + 2p(v_2|\omega_1| + v_1|\omega_2|) + p^2|\omega_1\omega_2|$$





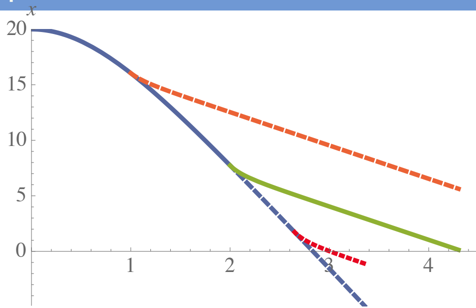
JAIS'14


x
20
15
10
5
0
1
2
3
4



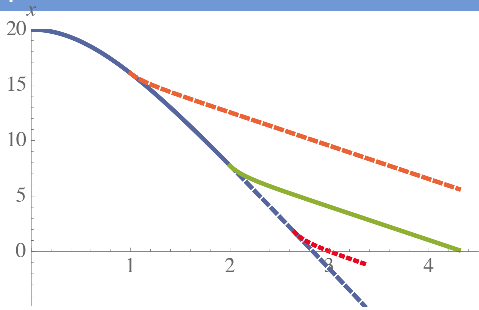
Example (Parachute)

$$\big((?(Q \land r = a) \cup r{:=}p); t{:=}0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \land x \geq 0 \land v < 0\}\big)^*$$



Example (Parachute)

$$\rightarrow \Big[\Big(\big(?(Q \wedge r = a) \cup r{:=}p\big); t{:=}0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \wedge x \geq 0 \wedge v < 0\}\Big)^*\Big]$$
$$(x = 0 \rightarrow v \geq m)$$



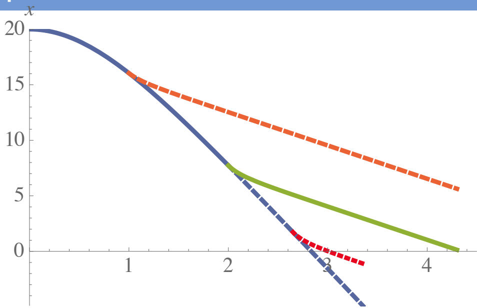
Example (Parachute)

$$\rightarrow \Big[\Big(\big(?(Q \land r = a) \cup r := p \big); t := 0; \{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \land x \geq 0 \land v < 0\} \Big)^* \Big] (x = 0 \rightarrow v \geq m)$$

Ex: Parachute Open or Keep Closed

$$Q \equiv v - gT > -\sqrt{g/p}$$

Conservatively bounded next velocity above parachute's limit velocity.



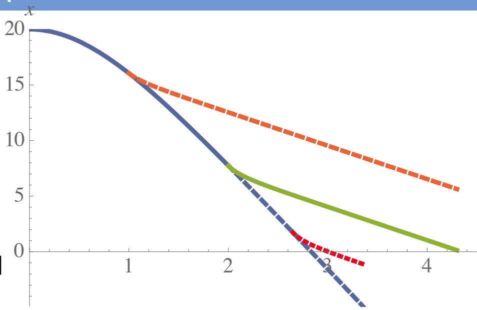
Example (Parachute)

$$m < -\sqrt{g/p} \rightarrow \Big[\Big(\big(?(Q \land r = a) \cup r := p\big); t := 0; \{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \le T \land x \ge 0 \land v < 0\}\Big)^*\Big] (x = 0 \rightarrow v \ge m)$$

$$Q \equiv v - gT > -\sqrt{g/p}$$

Conservatively bounded next velocity above parachute's limit velocity.
Limit by differential ghost:

$$y' = -\frac{p}{2}(v - \sqrt{g/p}) \qquad y^2\underbrace{(v+\sqrt{g/p})}_{>0} = 1$$



Example (Parachute)

$$m<-\sqrt{g/p} \rightarrow \Big[\big(((?(Q \land r = a) \cup r:=p); t:=0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \le T \land x \ge 0 \land v < 0\}\big)^*\Big]$$
$$(x = 0 \rightarrow v \ge m)$$

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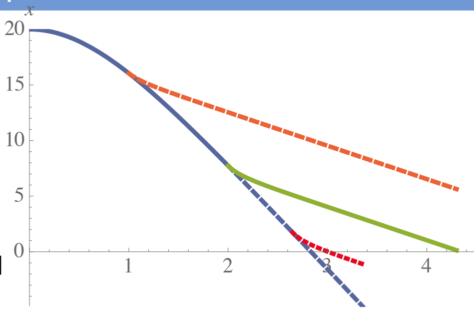
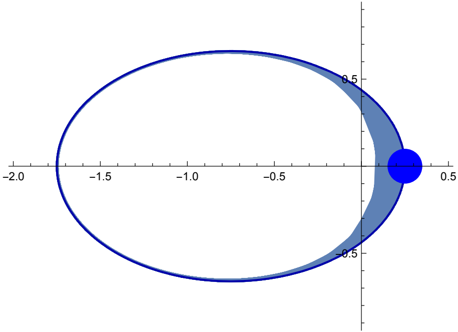


$v \geq \mathrm{old}(v) - gt$ if closed

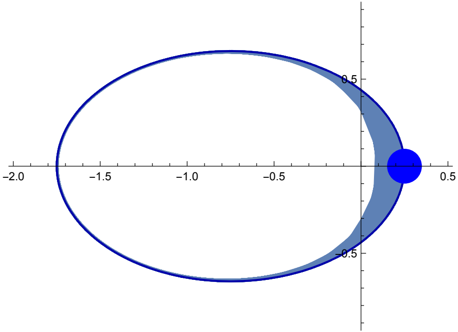
Example (Parachute)

$$m{<}-\sqrt{g/p} \rightarrow \Big[\Big(\big(?(Q \land r = a) \cup r{:=}p\big); t{:=}0;$$
$$\{x' = v, v' = -g + rv^2, t' = 1 \,\&\, t \leq T \land x \geq 0 \land v < 0\}\Big)^*\Big]$$
$$(x = 0 \rightarrow v \geq m)$$



−2.0
−1.5
−1.0
−0.5
0.5
0.5
−0.5

- $-\frac{x}{\sqrt{x^2+y^2}}$ opposite direction
- $\frac{1}{x^2+y^2}$ inverse-square law

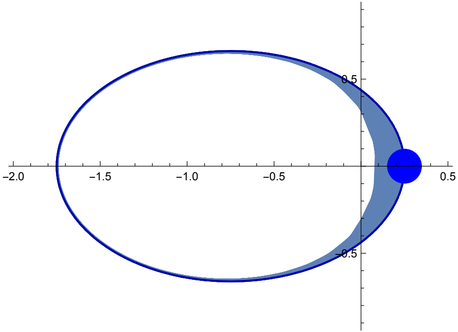


Example (Two Body Problem)

$$[x' = v, v' = -x/(x^2+y^2)^{3/2},$$
$$y' = w, w' = -y/(x^2+y^2)^{3/2}]$$

Ex: Kepler Two Body Problem

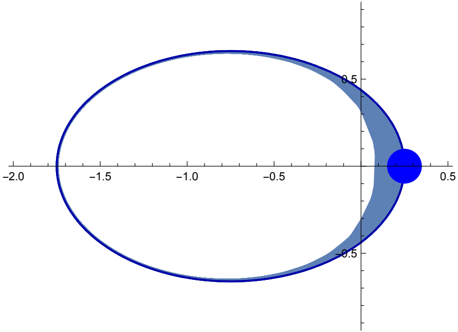
- $-\frac{x}{\sqrt{x^2+y^2}}$ opposite direction
- $\frac{1}{x^2+y^2}$ inverse-square law
- Energy preservation



Example (Two Body Problem)

$$\frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E \rightarrow$$
$$[x' = v, v' = -x/(x^2+y^2)^{3/2},$$
$$y' = w, w' = -y/(x^2+y^2)^{3/2}]\ \frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E$$

- $-\frac{x}{\sqrt{x^2+y^2}}$ opposite direction
- $\frac{1}{x^2+y^2}$ inverse-square law
- Energy preservation
- Well-definedness



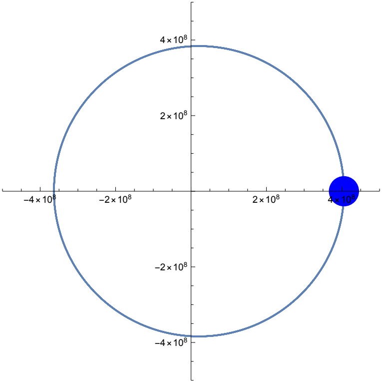
Example (Two Body Problem)

$$\frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E \rightarrow$$
$$[x' = v, v' = -x/(x^2+y^2)^{3/2},$$
$$y' = w, w' = -y/(x^2+y^2)^{3/2}] \; \frac{v^2+w^2}{2} - \frac{1}{\sqrt{x^2+y^2}} = E$$

$\& \, x \neq 0 \lor y \neq 0$

Exercise: Moon Gravitates Around the Earth

- G Gravitational constant $6.67430 * 10^{-11}$
- M Mass of the Earth
- m Mass of the Moon



Example (Moon around Earth)

$$\dots \rightarrow [x' = v, v' = -GMx/(x^2+y^2)^{3/2}, \\ y' = w, w' = -GMy/(x^2+y^2)^{3/2} \,\&\, x \neq 0 \lor y \neq 0] \dots$$

Summary: Proving ODEs

Differential Invariant

$$\frac{Q \rightarrow [x' := f(x)](P)'}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

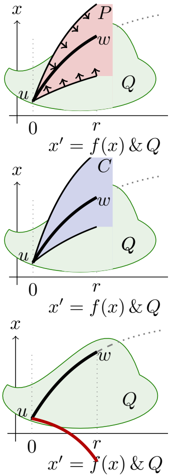
Differential Cut

$$\frac{P \rightarrow [x' = f(x) \,\&\, Q]C \quad P \rightarrow [x' = f(x) \,\&\, Q \land C]P}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

Differential Ghost

$$\frac{P \leftrightarrow \exists y\, G \quad G \rightarrow [x' = f(x), y' = g(x,y) \,\&\, Q]G}{P \rightarrow [x' = f(x) \,\&\, Q]P}$$

if $g(x,y) = a(x)y + b(x)$, so has long solution!



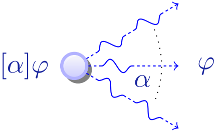
Springer'10, LMCS'12, LICS'12,JAR'17,LICS'18,JACM'20

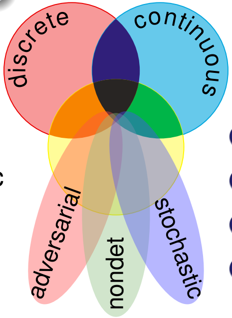
Outline (Logic for CPS)

Logical foundations make a big difference for CPS, and vice versa

differential dynamic logic

dL = DL + HP





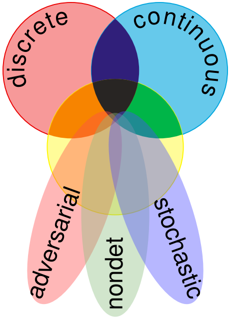
- Multi-dynamical systems
- Hybrid programs + dL logic
- Compositional proofs
- Decide invariant by dL

1. Analytic foundations
2. Practical proving
3. Significant applications
4. Bring sciences together

Programming CPS $\neq$ program cyber $\|$ program physics (mutual ignorance)

CPSs deserve proofs as safety evidence!

- Verified CPS implementations by ModelPlex FMSD'16
- Correct CPS execution PLDI'18
- CPS proof and tactic languages+libraries ITP'17
- Big CPS built from safe components STTT'18
- ODE invariance JACM'20
- ODE liveness FAC'21
- ODE stability TACAS'21
- Invariant generation FMSD'21
- Safe AI autonomy in CPS AAAI'18
- Refinement + system property proofs LICS'16
- CPS information flow LICS'18
- Hybrid games TOCL'15
- Constructive hybrid games IJCAR'20



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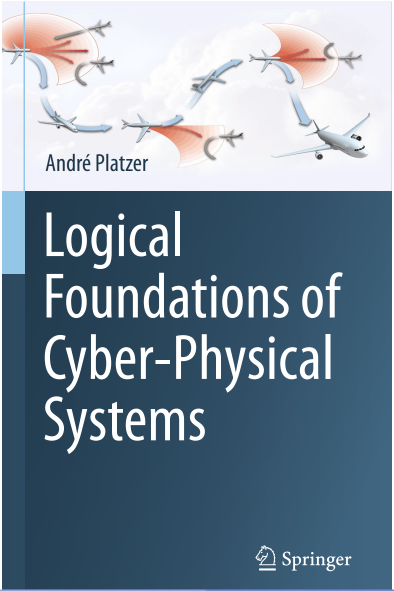

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