Karlsruhe Institute of Technology

Formale Systeme II: Theorie

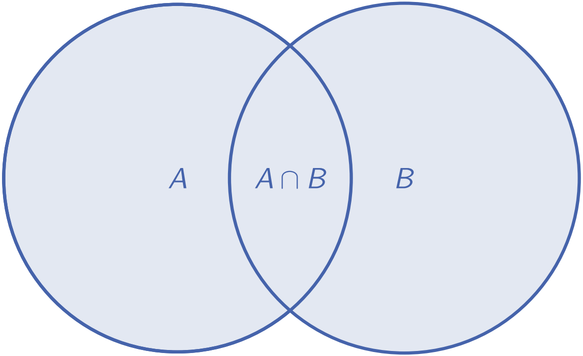
Axiomatic Set Theory

SS 2022

Prof. Dr. Bernhard Beckert · Dr. Mattias Ulbrich
Slides partially courtesy by Prof. Dr. Peter H. Schmitt

Motivation

Do you know set theory?



Do you know axiomatic set theory?

$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y.$$

$$\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z \neg(z \in x \wedge z \in y)).$$

$$\exists y \forall z(z \in y \leftrightarrow z \in x \wedge \phi(z)).$$

for any formula ϕ not containing y.

$$\exists y \forall x(x \notin y).$$

$$\exists y \forall x(x \in y \leftrightarrow x = z_1 \vee x = z_2).$$

$$\exists y \forall z(z \in y \leftrightarrow \forall u(u \in z \rightarrow u \in x)).$$

$$\exists y \forall z(z \in y \leftrightarrow \exists u(z \in u \wedge u \in x)).$$

$$\begin{aligned} \exists w(\emptyset \in w \wedge \quad & \forall x(x \in w \rightarrow \\ & \exists z(z \in w \wedge \\ & \forall u(u \in z \leftrightarrow u \in x \vee u = x)))). \end{aligned}$$

$$\begin{aligned} & \forall x, y, z(\psi(x, y) \wedge \psi(x, z) \rightarrow y = z) \rightarrow \\ & \exists u \forall w_1(w_1 \in u \leftrightarrow \exists w_2(w_2 \in a \wedge \psi(w_2, w_1))). \end{aligned}$$

$$\begin{aligned} & \forall x(x \in z \rightarrow x \neq \emptyset \wedge \\ & \forall y(y \in z \rightarrow x \cap y = \emptyset \vee x = y)) \\ & \rightarrow \\ & \exists u \forall x \exists v(x \in z \rightarrow u \cap x = \{v\}). \end{aligned}$$

Georg Cantor

Georg F.L.P. Cantor

	1845	born in St. Petersburg
	1862	studies in Zürich, Göttingen
-	1867	and Berlin
	1872	foundations of
-	1884	axiomatic set theory
	1869	Professor
-	1918	in Halle (Saale)
	1918	died in Halle

Georg Cantor

Pioneering Publications:

Über unendliche Punctmanichfaltigkeiten.
Math. Ann. 15(1879), 1–7, 17(1880), 355–358, 20(1882), 113–121, 21(1883), 51–58 and 545–586, 23(1884), 453–488

Beiträge zur Begründung der transfiniten Mengenlehre.
Math. Ann. 46(1895), 481–512, 49(1897), 207–245.

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‘Naive’ Set Theory

Beiträge zur Begründung der transfiniten Mengenlehre.

Von

GEORG CANTOR in Halle a./S.

(Erster Artikel.)

> „Hypotheses non fingo.“
>
> „Neque enim leges intellectui aut rebus damus ad arbitrium nostrum, sed tanquam scribae fideles ab ipsius naturae voce latas et prolatas excipimus et describimus.“
>
> „Veniet tempus, quo ista quae nunc latent, in lucem dies extrahat et longioris aevi diligentia.“

Göttinger Digitalisierungszentrum

§ 1.

Der Mächtigkeitsbegriff oder die Cardinalzahl.

Unter einer ‚Menge‘ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschauung oder unseres Denkens (welche die ‚Elemente‘ von M genannt werden) zu einem Ganzen.

In Zeichen drücken wir dies so aus:

(1) $$M = \{m\}.$$

Gottlob Frege

Friedrich Ludwig Gottlob Frege,
1848 – 1925

Logician, Mathematician, Philosopher
extra-ordinary professor in Jena

1879 “Begriffsschrift”:
“most important date in history of logic since Aristoteles”
Bedürftig, Murawski: Philosophie der Mathematik, 2010

Goal: Found mathematics on logical basis
Works: “Grundlagen der Arithmetik”, “Grungesetze der Arithmetik”

Contributions: Predicate logic as we know it today.

Frege's Begriffsschrift

Before Frege

- Ancient Logic: Collection of rigid rule schemata: *Syllogisms*
- Syllogisms were incomplete: Not every argument was possible
- Impossible: "Reductio ad absurdum" (Proof by contradiction)

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- the concept of (free) quantifiers,
- and a complete and correct calculus

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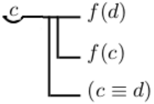
Notation very different from today:



instead of $\forall c.\ (c = d \to f(c) \to f(d))$

Antinomies

Bijection

Let A be a set. There is no bijection $f : A \to \mathbb{P}(A)$.

Proof (Diagonalisation). Assume f exists.
Choose $X = \{a \in A \mid a \notin f(a)\}$. Since $X \subseteq A$, there is $x \in A$ with $f(x) = X$. But both $x \in f(x)$ and $x \notin f(x)$ contradict themselves. $\lightning$

Cantor's Theorem

For every set A, its powerset $\mathbb{P}(A)$ has a larger cardinality.

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Cantor's Antinomy (1899)

The "set of all conceivable objects" cannot exist:
Its powerset (which is conceivable) would have to have larger cardinality.

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Antinomies

Bertrand Russell: Letter to G. Frege

"Sie behaupten [...] es könne auch die Funktion das unbestimmte Element bilden. Dies habe ich früher geglaubt, jedoch jetzt scheint mir diese Ansicht zweifelhaft, wegen des folgenden Widerspruchs: Sei w das Prädikat, ein Prädikat zu sein, welches von sich selbst nicht prädiziert werden kann. Kann man w von sich selbst prädizieren? Aus jeder Antwort folgt das Gegenteil. Deshalb muss man schließen, dass w kein Prädikat ist. [...] Daraus schließe ich, dass unter gewissen Umständen eine definierbare Menge kein Ganzes bildet."

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Let $R := \{x \mid x \notin x\}$. Now $R \in R$ is neither true nor false.

$\Longrightarrow$ Naive Set Theory is not consistent

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Insight:

A class term $\{x \mid \varphi(x)\}$ does not necessarily describe a set!

(Historical Sidenote)

Gottlieb Frege (1903):
***Grundgesetze der Arithmetik*, Nachwort**

"Einem wissenschaftlichen Schriftsteller kann kaum etwas Unerwünschteres begegnen, als daß ihm nach Vollendung einer Arbeit eine der Grundlagen seines Baues erschüttert wird. In diese Lage wurde ich durch einen Brief des Herrn Bertrand Russell versetzt, als der Druck dieses Bandes sich seinem Ende näherte."

Logicism (G. Frege)

Philosophical views:

What is the relationship betwen **logics** and **mathematics**?

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What is the relationship betwen **logics** and **mathematics**?

- Logical reasoning is a branch of mathematics.
  Mathematical subjects are “there” and wait to be described, formally captured.

Logicism (G. Frege)

Philosophical views:

What is the relationship betwen **logics** and **mathematics**?

- Logical reasoning is a branch of mathematics.
  Mathematical subjects are "there" and wait to be described, formally captured.

or

- Mathematics is an application of logical reasoning.
  1. There are valid axioms which are evidently true.
  2. All true propositions must be formally derived from axioms.

To the Rescue: Axiomatic Set Theory

Karlsruhe Institute of Technology

The dream of formal mathematics

- Leibniz (1646–1716): "Calculemus"
  (*Calculus ratiocinator* kann Wahrheitswert aller Aussagen berechnen)
- Frege: Mathematics as a logical theory
- Hilbert: Hilbert's Programme

Hilbert's Programme: secure foundations for all mathematics

- Consistent Axiomatisation
- Correct and Complete Calculus
- Proof of these Properties within this framework

$\Longrightarrow$ Axiomatic set theory (Zermelo-Fraenkel, Gödel-Bernaise, ...)

$\Longrightarrow$ End: Gödel's Incompleteness Theorems

This lecture

Zermelo-Fraenkel ...

- ... as a prominent first order theory
- ... as an example of modelling in FOL
- ... as foundations of mathematics

First Order Logic – Conservative Extension

Conservative Extension

Definition (proof-theoretic)

Let $\Sigma_1 \subseteq \Sigma_2$ be signatures, and T_i set of sentences in Fml_{Σ_i}.
T_2 is called a **conservative extension** of T_1 if

$$T_1 \models \varphi \iff T_2 \models \varphi \qquad \text{for all sentences } \varphi \in Fml_{\Sigma_1}$$

Conservative Extension

KIT Karlsruhe Institute of Technology

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Sufficient criterion (model-theoretic)

- Every model for T_1 can be extended to a model of T_2.

and

- Every restriction of a model of T_2 is a model of T_1.

Conservative Extension – Example

Let $\Sigma_0 = \{(0, s), (=), \alpha\}$

Axioms T0

- $\forall x.\ \neg s(x) = 0$
- $\forall x, y.\ s(x) = s(y) \rightarrow x = y$

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- Axioms T0
- $1 = s(0)$

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Axioms T2: $\Sigma_2 = \Sigma_0$

- Axioms T0
- $x = 0 \vee \exists y.x = s(y)$

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Axioms T0

- $\forall x.\ \neg s(x) = 0$
- $\forall x, y.\ s(x) = s(y) \to x = y$

Axioms T1: Σ_1 includes 1 — conservative extension of T0

- Axioms T0
- $1 = s(0)$

Axioms T2: $\Sigma_2 = \Sigma_0$ — **not** conservative extension of T1

- Axioms T0
- $x = 0 \vee \exists y. x = s(y)$

Conservativity Theorem

Theorem

Let Σ be a signature, T a Σ-theory and $\varphi(x, \bar{y})$ a Σ-formula.
Let $f \notin \Sigma$ be new function symbol

$$\text{If } T \models \forall \bar{y}.\ \exists x.\ \varphi(x, \bar{y})$$

then

$T \cup \{\forall y.\ \varphi(f(\bar{y}), y)$ is a conservative extension of T (over $\Sigma \cup \{f\}$)

Example

If $\forall y \exists x.\ y = x \cdot x$ is a theorem of some theory $R+$, then

function symbol *sqrt* can be added as conservative extension to $R+$ with definition $\forall y.\ y = sqrt(y) \cdot sqrt(y)$.

Zermelo-Fraenkel Axiom System

Zermelo and Fraenkel

Ernst Zermelo (1871-1953)

Abraham Fraenkel (1891-1965)

Zermelo and Fraenkel

Ernst Zermelo (1871-1953)

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1907 Zermelo proposes an axiom system with 7 axioms
1921 Fraenkel adds the replacement axiom
1930 Zermolo adds the foundation axiom
Axiom of choice was in initial set

Signature

$\Sigma = \{F, P, \alpha\}$ with

- $F = \emptyset$
- $P = \{\in, =\}$
- $\alpha(\in) = \alpha(=) = 2$

The semantics of equality is "built in", as usual.

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- $F = \emptyset$
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- $\alpha(\in) = \alpha(=) = 2$

The semantics of equality is “built in”, as usual.

That’s it. ...

Only two predicate symbols in the signature.
All other symbols often used $(\emptyset, \cup, \subset, \ldots)$ are derived symbols.

ZF

We will look at the axioms individually:

1. Original textual formulation, from
   Ernst Zermelo:
   Untersuchungen über die Grundlagen der Mengenlehre.
   In: Mathematische Annalen. 65 (1908)

2. As FOL formulas over the above signature, in modern notation

Zermelo-Fraenkel Axiom System

A1: Extensionality

„Ist jedes Element einer Menge M gleichzeitig Element der Menge N und umgekehrt [...], so ist immer $M = N$.

Oder kürzer: jede Menge ist durch ihre Elemente bestimmt."
[Zermelo, 1907]

$$\forall z. \, (z \in x \leftrightarrow z \in y) \rightarrow x = y$$

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- What about the converse implication?

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- Free variables in axioms are implicitly universally quantified.
- What about the converse implication?

  (Hint: Remember semantics of "="!)

Zermelo-Fraenkel Axiom System

A2: Foundation / Regularity

„Jede (rückschreitende) Kette von Elementen, in welcher jedes Glied Element des vorangehenden ist, bricht mit endlichem Index ab [...].

Oder, was gleichbedeutend ist: Jeder Teilbereich T enthält wenigstens ein Element t_0, das kein Element t in T hat." [Zermelo, 1930]

$$(\exists y.\ y \in x) \rightarrow \exists y.\ (y \in x \wedge \forall z.\ \neg(z \in x \wedge z \in y))$$

Zermelo-Fraengel Axiom System

A3: Separation Schema

,,Ist die Klassenaussage $F(x)$ definit* für alle Elemente einer Menge M, so besitzt M immer eine Untermenge M_F, welche alle diejenigen Elemente x von M, für welche $F(x)$ wahr ist, und nur solche als Elemente enthält."
$* \approx F(x)$ ist eine Formel. [Zermelo, 1908]

$$\exists y.\forall z.(z \in y \leftrightarrow z \in x \land \phi(z))$$

for any formula ϕ not containing y.

- is an axiom *schema*, contains a placeholder symbol

Zermelo-Fraengel Axiom System

A4: Empty set

„Es gibt eine (uneigentliche) Menge, die Nullmenge O, welche gar keine Elemente enthält." [Zermelo, 1908]

$$\exists y.\forall x.\neg(x \in y).$$

Zermelo-Fraenkel Axiom System

A5: Pair set

„[...]; sind a, b irgend zwei Dinge des Bereiches, so existiert immer eine Menge $\{a, b\}$, welche sowohl a als b, aber kein von beiden verschiedenes Ding x als Element enthält." [Zermelo, 1908]

$$\exists y.\forall x.(x \in y \leftrightarrow x = z_1 \lor x = z_2)$$

Zermelo-Fraenkel Axiom System

A6: Power set

„Jeder Menge T entspricht eine zweite Menge UT (die Potenzmenge von T), welche alle Untermengen von T und nur solche als Elemente enthält." [Zermelo, 1908]

$$\exists y.\forall z.(z \in y \leftrightarrow \forall u(u \in z \to u \in x))$$

Zermelo-Fraenkel Axiom System

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$$\exists y. \forall z. (z \in y \leftrightarrow \forall u (u \in z \rightarrow u \in x))$$

- Countable infinite set $\Longrightarrow$ uncountable power set expected.

Zermelo-Fraenkel Axiom System

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- Countable infinite set $\implies$ uncountable power set expected.
- Löwenheim-Skolem: There is a countable model of set theory.

Zermelo-Fraenkel Axiom System

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- Löwenheim-Skolem: There is a countable model of set theory.

$\Rightarrow$ Not all subsets can be guaranteed to exist

Zermelo-Fraenkel Axiom System

A7: Union / Sum

Jeder Menge T entspricht eine Menge GT (die Vereinigungsmenge von T), welche alle Elemente der Elemente von T und nur solche als Elemente enthält.

$$\exists y \forall z (z \in y \leftrightarrow \exists u (z \in u \wedge u \in y))$$

Zermelo-Fraenkel Axiom System

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$$GT = \bigcup T$$

Zermelo-Fraenkel Axiom System

A8: Infinity

Different Notion

„Der Bereich* enthält mindestens eine Menge Z, welche die Nullmenge als Element enthält und so beschaffen ist, daß jedem ihrer Elemente a ein weiteres Element der Form $\{a\}$ entspricht, oder welche mit jedem ihrer Elemente a auch die entsprechende Menge $\{a\}$ als Element enthält."

* Universum/Domäne [Zermelo, 1907]

$$\begin{aligned}\exists w.(\emptyset \in w \land \quad & \forall x(x \in w \to \\ & \quad \exists z(z \in w \land \\ & \qquad \forall u(u \in z \leftrightarrow u \in x \lor u = x))))\end{aligned}$$

Zermelo-Fraenkel Axiom System

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- $\emptyset \in Z \land \forall a. \, (a \in Z \to \{a\} \in Z)$

Zermelo-Fraenkel Axiom System

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$$\exists w.(\emptyset \in w \land \quad \forall x(x \in w \to \exists z(z \in w \land \forall u(u \in z \leftrightarrow u \in x \lor u = x))))$$

- $\emptyset \in Z \land \forall a.\ (a \in Z \to \{a\} \in Z)$
- $\emptyset \in Z \land \forall a.\ (a \in Z \to a \cup \{a\} \in Z)$

Zermelo-Fraenkel Axiom System

A9: Replacement

„Ist M eine Menge und wird jedes Element von M durch ein „Ding des Bereiches" [...] ersetzt, so geht M wiederum in eine Menge über." [Fraenkel, 1921]

$$\forall x, y, z.\ (\psi(x,y) \wedge \psi(x,z) \to y = z) \to \\ \exists u.\forall w_1.\ (w_1 \in u \leftrightarrow \exists w_2(w_2 \in a \wedge \psi(w_2, w_1)))$$

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- $$(\forall x.\ x \in u \to \exists y. \psi(x, y)) \land \\ (\forall x, y, z.\ (\psi(x, y) \land \psi(x, z) \to y = z)) \to \\ \exists v. \forall y. (y \in v \leftrightarrow \exists x.\ x \in u \land \psi(x, y))$$

Zermelo-Fraenkel Axiom System

A9: Replacement

„Ist M eine Menge und wird jedes Element von M durch ein „Ding des Bereiches" [...] ersetzt, so geht M wiederum in eine Menge über." [Fraenkel, 1921]

$$\forall x, y, z.\ (\psi(x,y) \wedge \psi(x,z) \rightarrow y = z) \rightarrow \\ \exists u. \forall w_1.\ (w_1 \in u \leftrightarrow \exists w_2 (w_2 \in a \wedge \psi(w_2, w_1)))$$

- $$(\forall x.\ x \in u \rightarrow \exists y. \psi(x,y)) \wedge \\ (\forall x, y, z.\ (\psi(x,y) \wedge \psi(x,z) \rightarrow y = z)) \rightarrow \\ \exists v. \forall y. (y \in v \leftrightarrow \exists x.\ x \in u \wedge \psi(x,y))$$
- ψ is function with dom $u \rightarrow \psi(u)$ is a set

Zermelo-Fraenkel Axiom System

A10 : Axiom of Choice

„Ist T eine Menge, deren sämtliche Elemente von 0 verschiedene Mengen und untereinander elementenfremd sind, so enthält ihre Vereinigung $\bigcup T$ mindestens eine Untermenge S_1, welche mit jedem Elemente von T ein und nur ein Element gemein hat.”
[Zermelo, 1907]

$$
\begin{array}{ll}
& \forall x(x \in z \to x \neq \emptyset \wedge \\
& \quad \forall y(y \in z \to x \cap y = \emptyset \vee x = y)) \\
\to & \\
& \exists u \forall x \exists v(x \in z \to u \cap x = \{v\})
\end{array}
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- "additional" axiom

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- "additional" axiom
- ZF versus ZFC

Conservative Extensions

A4: $\exists y.\forall x.\neg(x \in y)$

$$\text{new symbol } \emptyset \overset{\text{cons ex}}{\Longrightarrow} \forall x.\ \neg x \in \emptyset$$

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new symbol $\{\cdot,\cdot\} \overset{\text{cons ex}}{\Longrightarrow} \forall z_1, z_2, x.(x \in \{z_1, z_2\} \leftrightarrow x = z_1 \vee x = z_2)$

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A6: Powerset

new symbol $\mathbb{P}(\cdot) \stackrel{\text{cons ex}}{\Longrightarrow} \forall x, z.(z \in \mathbb{P}(x) \leftrightarrow \forall u.(u \in z \rightarrow u \in x))$

Class Terms

We will use for any formula $\phi(x)$ the syntactical construct

$$\{x \mid \phi(x)\},$$

called a class term.

Intuitively $\{x \mid \phi(x)\}$ is the collection of all sets a satisfying the formula $\phi(a)$.

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$\{x \mid \phi(x)\} \in \{y \mid \psi(y)\}$	is replaced by	$\exists u(\psi(u) \wedge \forall z(z \in u \leftrightarrow \phi(z)))$

Class Terms as Sets

A class term $\{x \mid \phi(x)\}$ does **not** necessarily denote a set.

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$$c \in c \Leftrightarrow c \notin c$$

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A class term $\{x \in A \mid \phi(x)\}$ **does** denote a set.

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Conservative extension: $\forall x, z.(z \in F_\phi(x) \leftrightarrow z \in x \wedge \phi(z))$

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Conservative extension: $\forall x, z.(z \in F_\phi(x) \leftrightarrow z \in x \land \phi(z))$

Different notation: $\forall x, z.(z \in \{t \in x \mid \phi(t)\} \leftrightarrow z \in x \land \phi(z))$

Some Abbreviations for Sets

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$\langle a, b \rangle$ is called the ordered pair of a and b.

Existence Claims

The following formulas follow from the ZF axioms

- $\exists x(x = \emptyset)$ A4 empty set axiom

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The following theorems are derivable in ZF:

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- If A is a non-empty class term, then there is a set c satisfying
  $$\forall z (z \in c \leftrightarrow \forall u (u \in A \rightarrow z \in u)) \qquad c = \bigcap A$$

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- $\exists y \forall z (z \in y \leftrightarrow \exists u (u \in a \wedge z \in u))$ $\quad y = \bigcup a$

- $\exists y \forall z (z \in y \leftrightarrow z \in a \wedge z \notin b)$ $\quad y = a \setminus b$

Proof of Existence of Intersections

Goal $\exists y \forall z(z \in y \leftrightarrow z \in a \land z \in b)$

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Start with the subset axiom A3

$$\exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi(z)).$$

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Replace

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yields

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as required

$$a \cap b = \{z \in a \mid z \in b\}$$

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Let a, b be sets.
We seek c with $\forall z(z \in c \leftrightarrow (z \in a \vee z \in b))$

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Karlsruhe Institute of Technology

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The pair axioms, A5, $\exists y \forall x(x \in y \leftrightarrow x = z_1 \vee x = z_2)$

guarantees the existence of the set $d = \{a, b\}$

Proof of Existence of Unions

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We seek c with $\forall z(z \in c \leftrightarrow (z \in a \vee z \in b))$

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The sum axiom, A7, $\exists y \forall z(z \in y \leftrightarrow \exists u(z \in u \wedge u \in x))$

yields the existence of a set c satisfying

$$\forall z(z \in c \leftrightarrow \exists u(z \in u \wedge u \in d))$$

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yields the existence of a set c satisfying

$$\forall z(z \in c \leftrightarrow \exists u(z \in u \wedge u \in d))$$

Substituting $d = \{a, b\}$ yields the claim.

Ordered Pairs

The following formula can be proved in ZF:

$$\forall x_1, x_2, y_1, y_2(\quad \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \leftrightarrow x_1 = y_1 \wedge x_2 = y_2 \quad)$$

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Proof

$\langle a_1, a_2\rangle = \langle b_1, b_2\rangle$

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Proof

$$\langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle \quad \Rightarrow \quad \bigcap \bigcap \langle a_1, a_2 \rangle = \bigcap \bigcap \langle b_1, b_2 \rangle$$

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Proof

$$\begin{aligned} \langle a_1, a_2\rangle = \langle b_1, b_2\rangle \quad &\Rightarrow \quad \bigcap\bigcap\langle a_1, a_2\rangle = \bigcap\bigcap\langle b_1, b_2\rangle \\ &\Rightarrow \quad \bigcap\bigcap\{\{a_1\}, \{a_1, a_2\}\} = \bigcap\bigcap\{\{b_1\}, \{b_1, b_2\}\} \end{aligned}$$

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&\Rightarrow \quad \bigcap\bigcap\{\{a_1\}, \{a_1, a_2\}\} = \bigcap\bigcap\{\{b_1\}, \{b_1, b_2\}\} \\
&\Rightarrow \quad \bigcap(\{a_1\} \cap \{a_1, a_2\}) = \bigcap(\{b_1\} \cap \{b_1, b_2\})
\end{aligned}$$

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Proof

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Case $a_1 = a_2$

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Proof

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Case $a_1 = a_2$ Note $a_1 = a_2 \Leftrightarrow b_1 = b_2$

Case $a_1 \neq a_2$

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$$\langle a_1, a_2\rangle = \langle b_1, b_2\rangle \Rightarrow$$

$$\bigcup(\bigcup\langle a_1, a_2\rangle \setminus \bigcap\langle a_1, a_2\rangle) = \bigcup(\bigcup\langle b_1, b_2\rangle \setminus \bigcap\langle b_1, b_2\rangle$$

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Case $a_1 = a_2$ — Note $a_1 = a_2 \Leftrightarrow b_1 = b_2$

Case $a_1 \neq a_2$

$$\langle a_1, a_2\rangle = \langle b_1, b_2\rangle \Rightarrow$$

$$\bigcup(\bigcup\langle a_1, a_2\rangle \setminus \bigcap\langle a_1, a_2\rangle) = \bigcup(\bigcup\langle b_1, b_2\rangle \setminus \bigcap\langle b_1, b_2\rangle$$

$$\Rightarrow \bigcup(\{a_1, a_2\} \setminus \{a_1\}) = \bigcup(\{b_1, b_2\} \setminus \{b_1\})$$

Ordered Pairs

The following formula can be proved in ZF:

$$\forall x_1, x_2, y_1, y_2(\quad \langle x_1, x_2\rangle = \langle y_1, y_2\rangle \leftrightarrow x_1 = y_1 \land x_2 = y_2 \quad)$$

Proof

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Relations and Functions

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  $rel(r) \equiv \forall x(x \in r \to \exists x_1, x_2(x = \langle x_1, x_2 \rangle))$

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Towards Macro Structures

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5. $\forall x(0 \in x \land \forall y(y \in x \to y^+ \in x) \to \mathbb{N} \subseteq x)$.

Proof of Axiom 4

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The foundation axiom, A2,

$$\exists y(y \in x) \to \exists y(y \in x \land \forall z \neg(z \in x \land z \in y)),$$

instantiated for $x = \{n, m\}$ yields

$$\exists y(y \in \{n, m\} \land \forall z(z \in \{n, m\} \to z \notin y)).$$

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Case $n = 0$
By definition $n^+ = \{0\}$.
Thus obviously $0 \in n^+$ and also $n^+ \in x$.

Transitive Sets

Definition

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In symbols

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Lemma

1. n is transitive for all $n \in \mathbb{N}$.
2. $\mathbb{N}$ is transitive.

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If $x = n$, then by definition $x \subseteq n^+$.

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If $n \in \mathbb{N}$ and by induction hypothesis $n \subseteq \mathbb{N}$

then also $n^+ = n \cup \{n\} \subseteq \mathbb{N}$.

The order relation on $\mathbb{N}$

Claim

The $\in$-relation is the smallest transitive relation r on $\mathbb{N}$ with $\langle n, n^+\rangle \in r$ for all n. i.e.

$$\forall n, m(n \in m \to \langle n, m\rangle \in r)$$

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Case $n = m$ We immediately have $\langle m, m^+\rangle \in r$.

Set Theoretic Properties of $\mathbb{N}$ (II)

The $<$-relation on $\mathbb{N}$ coincides with the $\in$-relation.

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Any natural number n is the set of all its predecessors, i.e.

$$n = \{m \mid m < n\}.$$

The Recursion Theorem

Let F be a function satisfying $\mathrm{rng}(F) \subseteq \mathrm{dom}(F)$ and let u be an element in $\mathrm{dom}(F)$.
Then there exists exactly one function f satisfying

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3. $f(n^+) = F(f(n))$ for all $n \in \mathbb{N}$

The assumptions $\text{rng}(F) \subseteq \text{dom}(F)$ and $u \in \text{dom}(F)$ are needed to make sure that all function applications of F are defined.

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Thus by the last Peano axiom induction axiom, we get

$$x = \mathbb{N}$$

i.e. $f = g$.

Idea of Existence Proof

Idea

$$H = \{h \mid func(h) \wedge h(0) = u \wedge \exists n(n \neq 0 \wedge \mathrm{dom}(h) = n \wedge \forall m(m^+ \in n \to h(m^+) = F(h(m))))\}$$

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\end{array}
$$

and

$$f = \bigcup H$$

Addition

for every $m \in \mathbb{N}$ there is a unique function add_m such that

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Apply the recursion theorem with $u = m$ and $F(x) = x^+$

Multiplication

for every $m \in \mathbb{N}$ there is a unique function $mult_m$ such that

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Karlsruhe Institute of Technology

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Apply the recursion theorem with $u = 0$ and $F_m(x) = add_m(x)$.

The Integers

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The idea is to reconstruct an integer

$$z$$

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Since $\langle 5, 7 \rangle$ and $\langle 8, 10 \rangle$ would both represent the same number, we have to use equivalence classes of ordered pairs instead of pairs themselves.

The Ordinal Numbers

Intention

An ordinal x is a set such that $(x, \in)$ is a well-ordered set.

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We will denote ordinals by lowercase Greek letter, α, β, ...

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3. The set of all natural numbers, traditionally denoted by the letter ω, is an ordinal.
4. If α is an ordinal, then every element $\beta \in \alpha$ is an ordinal.

Two Types of Ordinals

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2. An ordinal α
   such that for all β with $\beta \in \alpha$
   there is γ such that $\beta \in \gamma$ and $\gamma \in \alpha$
   is called a *limit ordinal*.

Representation Theorem

For every well-ordered set $(G, <)$ there is a unique ordinal α such that

$$(G, <) \cong (\alpha, \epsilon)$$

Examples of Ordinals

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1. $0, 0^+ = 1, 0^{++} = 2, \ldots, \mathrm{n}, \ldots$
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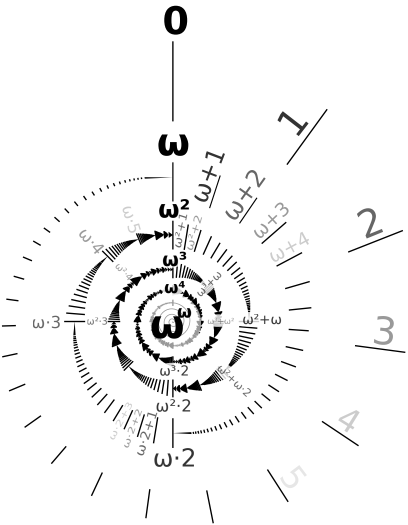
7. $\bigcup_{n\in\omega} \omega^n = \omega^\omega,$
8. $\omega^\omega,\ \omega^{\omega^\omega},\ \omega^{\omega^{\omega^\omega}},\ \ldots$

Visualisation



[found on wikipedia]

Limitations of ZFC

Gödel's Second Incompleteness Theorem

Assume T is a consistent theory which contains elementary arithmetic. Then $T \nvdash \mathit{Cons}(T)$; the consistency of T cannot be proved from T.

Continuuom Hypothesis independent from ZFC

There is no set whose cardinality lies strictly between that of the integers and the reals.