Formale Systeme II: Theorie

Dynamic Logic: Propositional Dynamic Logic

SS 2022

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Slides partially by Prof. Dr. Peter H. Schmitt

Requirements for this topic

- Fundamental knowledge of discreet structures (graphs, (equivalence) relations)
- General understanding of syntax and semantics of propositional and first order Logic
- General understanding of semantical concepts like satisfiability, decidability of logics

for instance from lecture "*Formale Systeme I*"

Dynamic Logic(s)

Overview – a family of logics

Modal Logics

↓

Propositional Dynamic Logic

↓

Dynamic Logic

↙ ↘

Hybrid DL   Java DL

Modal Logics: → Formal Systems I (recap here)
Java DL: Logic used in KeY
→ lecture "Formal Systems II – Applications"

Goals

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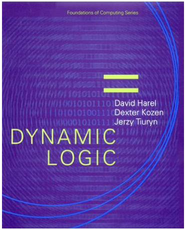
  - relative completeness
- concept of program verification on a while language
- logic for verification engines for realworld programming languages

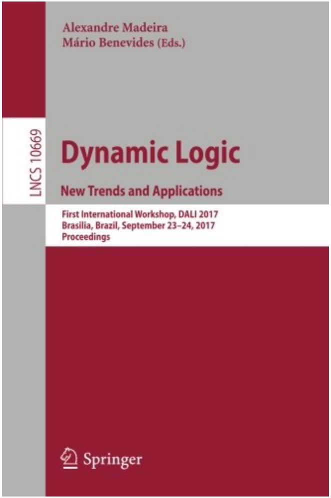
Literature

- *Formale Systeme II*
  Vorlesungsskript
  Peter H. Schmitt
  → Website

- *Dynamic Logic*
  Series: Foundations of Computing
  David Harel, Dexter Kozen and Jerzy Tiuryn
  MIT Press
  → Department Library



Still an Active Field ...



From the table of contents

- A Dynamic Logic for Learning Theory (Baltag et al.)
- Axiomatization and Computability of a Variant of Iteration-Free PDL with Fork (Balbiani et al.)
- Dynamic Preference Logic as a Logic of Belief Change (Souza et al.)
- Dynamic Logic: A Personal Perspective (**Vaughan Pratt**)
- ...

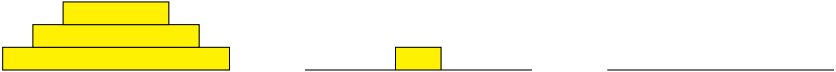
Motivating Example

Introductory Example

Introductory Example

The Towers of Hanoi

Introductory Example



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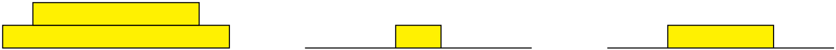

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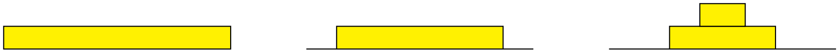

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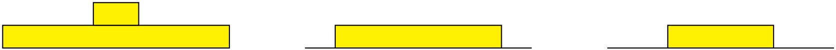

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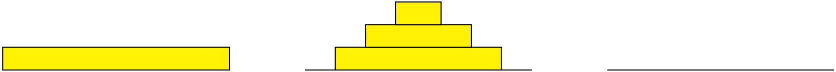

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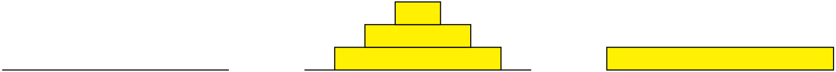

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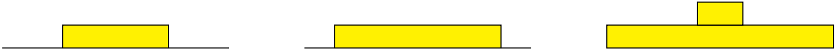

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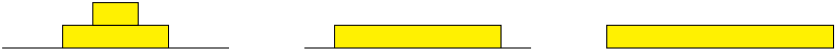

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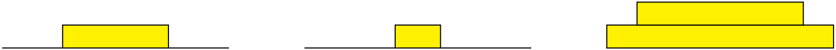

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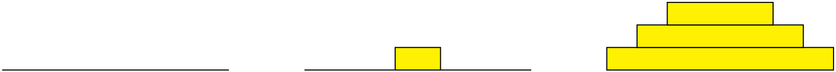

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improved:

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THAT IS DYNAMIC LOGIC

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Syntax/semantics of dynamic logic build on top of modal logic.

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Kripke Structure (S, R, I):

- Given a signature Σ
- Kripke Frame (S, R)
- Interpretation $I : S \to 2^{\Sigma}$

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For a signature Σ and Kripke structure (S, R, I)

$I, s \models \varphi \iff$ Formula φ holds in state $s \in S$
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This means:

$$\begin{array}{lcl} I, s \models \varphi \wedge \psi & \iff & I, s \models \varphi \text{ and } I, s \models \psi \\ I, s \models \varphi \vee \psi & \iff & I, s \models \varphi \text{ or } I, s \models \psi \\ I, s \models \varphi \rightarrow \psi & \iff & I, s \models \varphi \text{ implies } I, s \models \psi \\ I, s \models \neg\varphi & \iff & \text{not } I, s \models \varphi \end{array}$$

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Karlsruhe Institute of Technology

Applications of modal logics

Logics of *necessity* and *possibility* – philosophy.

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- Extends modal logic

More than one modality

Karlsruhe Institute of Technology

Multi-modal logic

Have different Box operators with different accessibility relations:

$$\Box_\alpha, \Box_\beta, \Box_\gamma, \ldots$$

($\rightarrow$ basic actions ins "Towers of Hanoi")

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Propositional Dynamic Logic (PDL):

- Signature Σ of propositional variables
- Set $A = \{\alpha, \beta, \ldots\}$ of atomic actions/programs
- We write $[\alpha]$ instead of $\Box_\alpha$

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Regular Programs =
Regular Expressions over atomic programs and tests

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4. $P \in \Pi_{\Sigma,A}, \varphi \in Fml_{\Sigma,A}^{PDL} \implies [P]\varphi, \langle P \rangle \varphi \in Fml_{\Sigma,A}^{PDL}$

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4. $P \in \Pi_{\Sigma,A}, \varphi \in Fml^{PDL}_{\Sigma,A} \implies [P]\varphi, \langle P \rangle \varphi \in Fml^{PDL}_{\Sigma,A}$

Programs and Formulae are mutually dependent definitions and must be seen simultaneously.

PDL Formulas – Examples

KIT Karlsruhe Institute of Technology

→ Towers of Hanoi

$A = \{moveS, moveO\}, \qquad \Sigma = \{S1\}$

$$S1 \to \langle (moveO)^* \,;\, moveS \rangle \neg S1$$

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multi-level and nested modalities

$A = \{\alpha, \beta\}, \qquad \Sigma = \{P, Q\}$

$$[\alpha \cup (\mathbf{?}P \,;\, \beta)^*]Q$$

$$[\alpha]P \rightarrow [\alpha^*]P$$

$$[\alpha]\langle\beta\rangle\big(P \rightarrow [\alpha^*]Q\big)$$

$$[\alpha \,;\, \mathbf{?}\langle\beta\rangle P \,;\, \beta]Q$$

PDL – Semantics

Given a signature Σ and atomic programs A

(multi-modal propositional) Kripke frame (S, ρ)

- set of states S
- function $\rho : A \to 2^{S \times S}$ accessibility relations for atomic programs

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Kripke structure (S, ρ, I)

- Kripke frame (S, ρ)
- interpretation $I : S \to 2^{\Sigma}$

$\Rightarrow$ same as for modal logic

PDL – Program Semantics

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from $\rho : A \to 2^{S^2}$ to $\rho : \Pi_{\Sigma,A} \to 2^{S^2}$

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\rho(\pi_1 \,;\, \pi_2) & = \rho(\pi_1) \,;\, \rho(\pi_2) \\
& = \{(s, s') \mid \text{ex. } t \text{ with } (s,t) \in \rho(\pi_1) \text{ and } (t, s') \in \rho(\pi_2)\}
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$$\begin{aligned}\rho(\pi^*) &= \mathrm{rtcl}(\rho(\pi)) = \textstyle\bigcup_{n=0}^{\infty} \rho(\pi)^n \qquad \textit{refl. transitive closure} \\ &= \{(s_0, s_n) \mid \text{ex. } n \text{ with } (s_i, s_{i+1}) \in \rho(\pi) \text{ for } 0 \le i < n\}\end{aligned}$$

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&\rho(\mathbf{?}F) && = \{(s, s) \mid I, s \models F\}
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$$

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For a signature Σ, basic programs A and Kripke structure (S, ρ, I)

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$$I, s \models \langle\pi\rangle\varphi \quad \iff \quad I, s' \models \varphi \text{ for some } s' \in S \text{ with } (s, s') \in \rho(\pi)$$

Tautologies

Dual operators

$$[\pi]\varphi \;\leftrightarrow\; \neg\langle\pi\rangle\neg\varphi$$

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- all tautologies for modal logic **K**

A Calculus for Propositional Dynamic Logic

Axioms

All propositional tautologies

$[\pi](\varphi \to \psi)$	$\to$	$([\pi]\varphi \to [\pi]\psi)$	(ML1 = K)
$[\pi](\varphi \land \psi)$	$\leftrightarrow$	$[\pi]\varphi \land [\pi]\psi$	(ML2)
$[\pi_1; \pi_2]\varphi$	$\leftrightarrow$	$[\pi_1][\pi_2]\varphi$	(PDL1)
$[\pi_1 \cup \pi_2]\varphi$	$\leftrightarrow$	$[\pi_1]\varphi \land [\pi_2]\varphi$	(PDL2)
$[?\varphi]\psi$	$\leftrightarrow$	$\varphi \to \psi$	(PDL3)
$[\pi^*]\varphi$	$\leftrightarrow$	$\varphi \land [\pi][\pi^*]\varphi$	(PDL4)
$\varphi \land [\pi^*](\varphi \to [\pi]\varphi)$	$\to$	$[\pi^*]\varphi$	(IND)

Rules

$$\frac{\varphi, \; \varphi \to \psi}{\psi} \qquad \text{(MP)}$$

$$\frac{\varphi}{[\pi]\varphi} \qquad \text{(GEN)}$$

Theorem

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The presented calculus is sound and complete.

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Proof

See e.g.,pp. 559-560
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or

D. Harel, D. Kozen and J. Tiuryn
Dynamic Logic
in *Handbook of Philosophical Logic*, 2^{nd} *edition* , *volume 4*
by Kluwer Academic Publisher, 2001.

Higher level program constructors

Syntactic Sugar

- PDL syntax has elementary program operators
- Enrich it by defining new operators ("macros")

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$$\begin{aligned} \text{skip} &:= \ \mathbf{?}\mathit{true} \\ \text{fail} &:= \ \mathbf{?}\mathit{false} \\ \text{if } \varphi \text{ then } \alpha \text{ else } \beta &:= \ (\mathbf{?}\varphi\, ;\, \alpha) \cup (\mathbf{?}\neg\varphi\, ;\, \beta) \end{aligned}$$

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\text{skip} &:= \mathbf{?}\textit{true} \\
\text{fail} &:= \mathbf{?}\textit{false} \\
\text{if } \varphi \text{ then } \alpha \text{ else } \beta &:= \big(\mathbf{?}\varphi\,;\alpha\big) \cup \big(\mathbf{?}\neg\varphi\,;\beta\big) \\
\text{while } \varphi \text{ do } \alpha &:= (\mathbf{?}\varphi\,;\alpha)^*\,;\mathbf{?}\neg\varphi
\end{aligned}$$

More PDL Tautologies

$$[\mathsf{skip}]\varphi \quad \leftrightarrow \quad \varphi$$

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Is PDL decidable?

$$\iff$$

Is there an algorithm that terminates on every input and computes whether a PDL-formula $\phi \in Fml^{PDL}_{\Sigma,A}$ is satisfiable.

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Answer:

YES, PDL is decidable!

Fischer and Ladner (1979)

General Idea:

$\varphi \in \mathit{Fml}^{PDL}$ has a model $\iff$ φ has a model of bounded size.

For every Kripke structure, a bounded Kripke structure can be defined which is indistinguishable for φ.

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For every Kripke structure, a bounded Kripke structure can be defined which is indistinguishable for φ.

Preliminary lemma: Decidability for modal logic

The proof idea is the same, yet simpler.

Fischer-Ladner Closure

Reduced syntax

Only connectors $\rightarrow$, *false*, $\Box$ are allowed $\Rightarrow$ simplifies proofs.

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assigns to φ the set of subformulas of φ.

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$$\begin{aligned}
FL^{mod}(\varphi \rightarrow \psi) &= \{\varphi \rightarrow \psi\} \cup FL^{mod}(\varphi) \cup FL^{mod}(\psi) \\
FL^{mod}(\mathit{false}) &= \{\mathit{false}\} \\
FL^{mod}(p) &= \{p\} \qquad p \in \Sigma \\
FL^{mod}(\Box\varphi) &= \{\Box\varphi\} \cup FL^{mod}(\varphi)
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\end{aligned}$$

Observation

$$|FL^{mod}(\varphi)| \leq |\varphi|$$

Filtration for modal logic

Filtration

For a Kripke structure S, R, I define a bounded structure $\widetilde{S}, \widetilde{R}, \widetilde{I}$ with

$$S, R, I, s \models \varphi \iff \widetilde{S}, \widetilde{R}, \widetilde{I}, \widetilde{s} \models \varphi$$

Filtration for modal logic

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Central Idea

States are **undistinguishable** for φ if they are equal on $FL^{mod}(\varphi)$.

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$$s \equiv t \iff (I, s \models \psi \Leftrightarrow I, t \models \psi \text{ for all } \psi \in FL^{mod}(\varphi))$$

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Central Idea

States are **undistinguishable** for φ if they are equal on $FL^{mod}(\varphi)$.

$$s \equiv t \iff (I, s \models \psi \Leftrightarrow I, t \models \psi \text{ for all } \psi \in FL^{mod}(\varphi))$$

$$\widetilde{s} := \{s' \mid s' \equiv s\} \qquad \ldots \text{ equivalence classes}$$

$$\widetilde{S} := \{\widetilde{s} \mid s \in S\}$$

$$\widetilde{R} := \{(\widetilde{s}, \widetilde{s'}) \mid (s, s') \in R\}$$

$$\widetilde{I}(\widetilde{s}) := I(s)$$

Fischer-Ladner Filtration

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\widetilde{s} &:= \{s' \mid s' \equiv s\} \\
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Lemma

$$|\widetilde{S}| \;\leq\; 2^{|FL^{mod}(\varphi)|} \;\leq\; 2^{|\varphi|}$$

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$$|\widetilde{S}| \;\leq\; 2^{|FL^{mod}(\varphi)|} \;\leq\; 2^{|\varphi|}$$

Lemma (proved by structural induction)

$$S, R, I, s \models \varphi \iff \widetilde{S}, \widetilde{R}, \widetilde{I}, \widetilde{s} \models \varphi$$

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$$S, R, I, s \models \varphi \iff \widetilde{S}, \widetilde{R}, \widetilde{I}, \widetilde{s} \models \varphi$$

Theorem (*small model property*)

For any PDL formula φ it can be decided if φ is satisfiable by inspecting a finite number (those up to size $2^{|\varphi|}$) of models.

Fischer-Ladner Closure for PDL

Operator

$$FL : Fml^{PDL} \to 2^{Fml^{PDL}}$$

$FL(\varphi)$ smallest set satisfying

1	$\varphi \in FL(\varphi)$		
2	$(\psi_1 \to \psi_2) \in FL(\varphi)$	$\Rightarrow$	$\psi_1 \in FL(\varphi)$ and $\psi_2 \in FL(\varphi)$
3	$[\pi]\psi \in FL(\varphi)$	$\Rightarrow$	$\psi \in FL(\varphi)$
4	$[\pi_1;\pi_2]\psi \in FL(\varphi)$	$\Rightarrow$	$[\pi_1][\pi_2]\psi \in FL(\varphi)$
5	$[\pi_1 \cup \pi_2]\psi \in FL(\varphi)$	$\Rightarrow$	$[\pi_1]\psi \in FL(\varphi)$ and $[\pi_2]\psi \in FL(\varphi)$
6	$[\pi^*]\psi \in FL(\varphi)$	$\Rightarrow$	$[\pi][\pi^*]\psi \in FL(\varphi)$
7	$[\mathbf{?}\psi_1]\psi_2 \in FL(\varphi)$	$\Rightarrow$	$\psi_1 \in FL(\varphi)$

Lemma (not obvious)

$$|FL(\varphi)| \leq |\varphi|$$

Fischer-Ladner Filtration

Same construction as for modal logic

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Corollary

PDL has the small model property:
If $\varphi \in Fml^{PDL}$ is satisfiable, it has a model with at most $2^{|\varphi|}$ states.

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Naive approach used for proof

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Complexity of Deciding PDL

The decision problem for PDL is in EXPTIME:
can be decided by a deterministic algorithm in $O(2^{p(n)})$ for some polynomial p.

⇝[Harel et al. Ch. 8]

Deduction Theorem and Compactness

Logical Consequence

$M \subseteq Fml^{PDL}, \quad \varphi \in Fml^{PDL}$

Global Consequence

$M \models^G \varphi :\Longleftrightarrow$
for all Kripke structures (S, ρ, I):
$I, s \models M$ for all $s \in S$ implies $I, s \models \varphi$ for all $s \in S$

Local Consequence

$M \models^L \varphi :\Longleftrightarrow$
for all Kripke structures (S, ρ, I):
for all $s \in S$: $I, s \models M$ implies $I, s \models \varphi$

Local consequence is stronger: $M \models^L \varphi \quad \begin{matrix} \Longrightarrow \\ \not\Longleftarrow \end{matrix} \quad M \models^G \varphi$

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Lemma

$$\psi \models^{G} \varphi \iff \models ([(\beta_1 \cup \ldots \cup \beta_k)^*]\psi) \to \varphi$$

with $B := \{\beta_1, ..., \beta_k\}$ the atomic programs occurring in ψ, φ.

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4. $S^-(s), s \models \alpha \iff S, s \models \alpha$ for all formulas α over B

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Decidable:

The consequence problem $\psi \models^G \varphi$ **is** decidable for PDL.

Compactness of PDL

Recall: Compactness Theorem

$$M \models^G \varphi \quad \Longleftrightarrow \quad \text{exists finite } E \subseteq M \text{ with } E \models^G \varphi$$

Holds for:

Propositional Logic, First Order Logic, **not** for higher order logic

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PDL is not compact

because it has transitive closure "built in".

Deducibility Problem in PDL

Quote:

> *[T]he problem of whether an arbitrary PDL formula p is deducible from a single fixed axiom scheme is of extremely high degree of undecidability, namely* Π^1_1*-complete.*

Meyer, Streett, Mirkowska:
The Deducibility Problem in Propositional Dynamic Logic, 1981

Variants and Conclusion

Variant: Converse Programs

Karlsruhe Institute of Technology

Idea: Add actions reverting action effects

Add further program constructor $\cdot^{-1}$:
$\pi \in \Pi \implies \pi^{-1} \in \Pi$
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Axiom schemes: for all $\varphi \in \mathit{Fml}^{PDL}$, $\pi \in \Pi$

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Complete

Adding the axioms to the known PDL calculus gives a correct and complete calculus for PDL with Converse.

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Idea: Go beyond regular programs

Instead of regular programs, allow context-free grammar

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Expressiveness

Without fixed semantics of $\mathbb{N}$, recursion is strictly more expressive than looping.

State Vector Semantics

A propositional Kripke structure $\mathcal{K} = (S, \rho, I)$ is determined by:

S	the set of states
$\rho : A \to S \times S$	the accessibility relations for atomic programs e
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We call this the state vector semantics.

- Strictly larger set of tautologies.
- Obviously decidable.
- Evaluation of propositional variables fixes the state (and the accessibility of successor states)

Lemma

Let

- $A = \{a_1, \ldots, a_k\}$

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Then

$$\langle \pi_{all} \rangle (state_U \wedge F) \rightarrow [\pi_{all}](state_U \rightarrow F)$$

is true in all state vector Kripke structures.

Theorem

Let H be the set of all formulas

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Then:

1. $\{F\} \cup H$ is satisfiable iff F is state vector satisfiable.
2. $H \models F$ iff $\models_{sv} F$.

Propositional Dynamic Logic – Summary

- extension of modal logic
- abstract notion of actions / atomic logic statements
- regular programs, with non-deterministic choice and Kleene-interation
- correct and complete calculus for tautologies
- satisfiability is decidable (in EXPTIME)
- logic is not compact
- deducibility is utterly undecidable
- deduction theorem can be rescued

Detection of dynamic execution errors in

IBM system automation's rule-based expert system

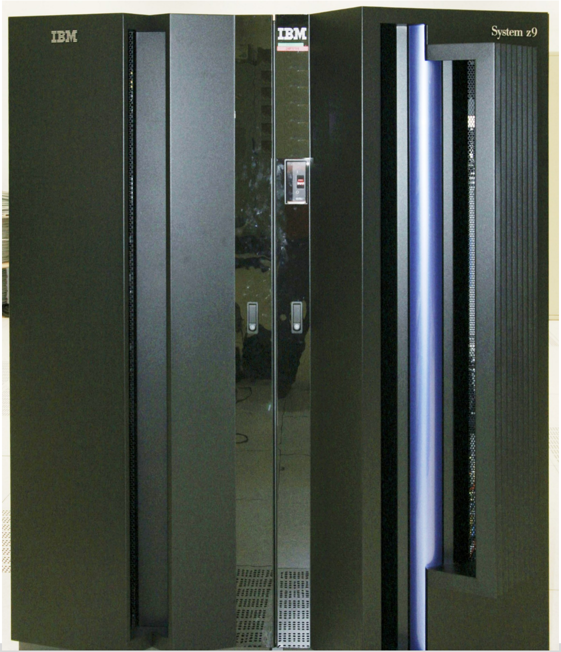
An Application of PDL

Reference

[SinzEtAl02]

Carsten Sinz, Thomas Lumpp, Jürgen Schneider, and Wolfgang Küchlin:
Detection of dynamic execution errors in IBM System Automation's rule-based expert system.
Information and Software Technology, 44(14):857–873, November 2002.

Context



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IBM zSeries

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System Automation

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Example

Flight booking center: 100s of users, many parallel apps

Example Rule

```
correlation set/status/compound/satisfactory :
when        status/compound NOT E {Satisfactory}
        AND status/startable E {Yes}
        AND ( ( status/observed E {Available, WasAvailable}
                AND status/desired E {Available}
                AND status/automation E {Idle, Internal}
                AND correlation/external/stop/failed E {false}
              )
              OR
              ( status/observed E {SoftDown, StandBy}
                AND status/desired E {Unavailable}
                AND status/automation E {Idle, Internal}
              )
            )
then    SetVariable status/compound = Satisfactory
        RecordVariableHistory status/compound
```

Fig. 4. Example of a correlation rule.

(taken from [SinzEtAl02])

Rules

when *cond* **then** $var = d$

- **AND**, **OR**, **NOT** allowed in conditions
- *var* **E** $\{ d_1, \ldots, d_2 \}$ – “element of”
- the **then** part can be executed if **cond** is true

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- Axiom $[\alpha_{var,d}]P_{var,d}$

Logical Encoding

Karlsruhe Institute of Technology

Semantics of a rule as program:

$$?\mathit{when}\,;\,\mathit{then}$$

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Semantics of all rules as program:

$$R := \big((\textbf{?}\mathit{when}_1\,;\,\mathit{then}_1) \cup \ldots \cup (\textbf{?}\mathit{when}_r\,;\,\mathit{then}_r)\big)^*$$

Proof Obligations

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under assumption of a precondition PRE

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Absence of Oscillation:
modelled using an extension of PDL with non-termination operator

Verification Experiment

Verification Technique

- state vector semantics
- translation of PDL to boolean SAT
- solving using SAT solver (Davies-Putnam)

Experiment:

- $\sim$40 rules
- resulted in $\sim$1500 boolean variables
- SAT solving < 1 sec

!! **violations found – before deployment**